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A Generalized Exponential Dual to Ratio type Imputation Method in Double Sampling Scheme for Estimation of Population mean with two Auxiliary Variables

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Abstract
The problem of missing values in the sample is very common in most surveys. Through this manuscript, we suggest a generalized exponential chain dual-to-ratio type imputation method and the corresponding point estimator of the finite population mean in a double sampling scheme with two auxiliary variables in case of missing data in the sample. The $Bias$ and $Mean\ Square\ Error$ (MSE) of the proposed estimator are derived in terms of parameters. Theoretical studies are carried out to verify the supremacy of the proposed point estimator over other existing estimators considered in this study. An empirical study has been done to compare the percent relative efficiency of the proposed class of estimators with other existing estimators by using one natural population data and another artificially generated population data, generated by Shukla and Thakur (2008) A simulation study have been also carried out among the proposed and existing estimators for numerical calculation of bias, mean square error and percent relative efficiencies. The numerical calculations are found similar in both the Empirical and simulation studies. The novelty of the study is that the proposed imputation method is more efficient than the existing imputation methods considered in this study.

Keywords: $Bias$, MSE, $Exponential\ estimators$, $Imputation$, $Simple\ random\ sampling\ and\ Double\ sampling.$

1 Introduction

Ratio type estimators are generally used for estimating the finite population mean of the study variable Y when the study variable is highly correlated with another variable(s) known as auxiliary variable. If the population mean of the auxiliary variable $\overline{X}$ is unknown then double sampling design is adopted to get an estimate of $\overline{X}$ with the aid of sample mean of a preliminary large sample on which only the variable X is measured. Cochran (1940) introduced an estimator of population mean by using the information of auxiliary variable in simple random sampling, known as classical ratio estimator or ratio estimator. Bahl and Tuteja (1991) proposed exponential ratio and exponential product type estimators in simple random sampling. The work is further extended by Kumar and Bahl (2006), Singh and Vishwakarma (2007), Singh and Choudhary (2012) and many others under simple random sampling. It is often seen that the auxiliary variable X is highly correlated with another variable Z which is relatively less correlated to the main variable Y. In such a situation the information of Z is used to estimate $\overline{X}$ in first phase sample. Chand (1975) introduced a technique of chaining the information of two auxiliary variables with the main variable Y. Further, similar works were extended by Mukerjee et al. (1987), Srivastava et al.

(1989), Singh and Choudhury (2012) and Kalita et al. (2013).
The above estimators are applicable to estimate the finite population mean of the study variable under the assumption that information on the main variable as well as auxiliary variable(s) are available in the sample(s). In practice, sometimes, above assumption is not met and non- response occurs. The imputation method is a standard technique to overcome the problems of incompleteness in data set and as such missing data are replaced by a functional value of main variable and auxiliary variable(s). Kalton et al. (1981) and Sande (1979) proposed imputation techniques to make incomplete data set structurally complete. Lee and Sarndal (1994) introduced Ratio method of imputation by using one auxiliary variable for estimating population mean in simple random sampling. Singh and Horn (2000) introduced compromised method of imputation and corresponding point estimator of population mean in simple random sampling. Similar work has been extended by Ahmad (2006), Lee et al. (1994, 1995), Shukla (2002), Shukla and Thakur (2008), Kadilar and Cingi (2008), Shukla et al. (2009), Thakur et al. (2011), Shukla et al. (2013), Thakur et al. (2013), Singh et al. (2014), Singh et al. (2015), Singh and Gogoi (2018), Nath and Singh (2018), Nath et al. (2020) Kumar et al. (2017) Thakur and Shukla (2022) and many others under simple random sampling and double sampling scheme.

2 Notations

Let $\Omega = \{1, 2, 3, \ldots \text{N}\}$ be a finite population of size N and Y be the study variable and X, Z are the auxiliary variables. $\overline{Y}, \overline{X}$ and $\overline{Z}$ are the population mean of the variables Y, X and Z respectively and let $\overline{X}$ be unknown. Consider a first phase sample S_1 of size n_1 is drawn from the population Ω using the SRSWOR method and a second sample S of size n is drawn from S_1. Let the second sample S contain $r(< n)$ responding units forming a subspace A and $r^* = n - r$ non-responding units with subspace A^c, such that $A \cup A^c = S$. For every $i \in A$, y_i is observed and available. For $i \in A^c$, y_i values are missing and the imputed values are computed. The i^{th} value x_i and z_i of the auxiliary variables are used as a source of imputation for missing data when $i \in A^c$ assuming that in S_1, the data $\{(x_i, z_i); i \in S\}$ and $\{(x_i, z_i); i \in S_1\}$ are known.

Different statistic and parameters are described in the following expressions

$\overline{x}_n = \frac{1}{n} \sum\limits_{i \in S} X_i$ $\qquad$ $\overline{y}_n = \frac{1}{n} \sum\limits_{i \in S} Y_i$ $\qquad$ $\overline{z}_n = \frac{1}{n} \sum\limits_{i \in S} Z_i$

$\overline{x}_r = \frac{1}{r} \sum\limits_{i \in A} X_i$ $\qquad$ $\overline{y}_r = \frac{1}{r} \sum\limits_{i \in A} Y_i$ $\qquad$ $\overline{z}_r = \frac{1}{r} \sum\limits_{i \in A} Z_i$

$\overline{x}_1 = \frac{1}{n_1} \sum\limits_{i \in S_1} X_i$ $\qquad$ $\overline{y}_1 = \frac{1}{n_1} \sum\limits_{i \in S_1} Y_i$ $\qquad$ $\overline{z}_1 = \frac{1}{n_1} \sum\limits_{i \in S_1} Z_i$

$\overline{X} = \frac{1}{N} \sum\limits_{i \in \Omega} X_i$ $\qquad$ $\overline{Y} = \frac{1}{N} \sum\limits_{i \in \Omega} Y_i$ $\qquad$ $\overline{Z} = \frac{1}{N} \sum\limits_{i \in \Omega} Z_i$

$\rho_{YX = \frac{S_{YX}}{S_X S_Y}}$, $\qquad$ $\rho_{YZ = \frac{S_{YZ}}{S_Y S_Z}}$, $\qquad$ $\rho_{XZ = \frac{S_{XZ}}{S_X S_Z}}$

$S_{YX} = \frac{1}{N-1} \sum\limits_{i \in \Omega} (X_i - \overline{X})(Y_i - \overline{Y})$ $\qquad$ $S_{YZ} = \frac{1}{N-1} \sum\limits_{i \in \Omega} (Y_i - \overline{Y})(Z_i - \overline{Z})$

$S_{XZ} = \frac{1}{N-1} \sum\limits_{i \in \Omega} (X_i - \overline{X})(Z_i - \overline{Z})$ $\qquad$ $S_X^2 = \frac{1}{N-1} \sum\limits_{i \in \Omega} (X_i - \overline{X})^2$

$S_Y^2 = \frac{1}{N-1} \sum\limits_{i \in \Omega} (Y_i - \overline{Y})^2$ $\qquad$ $S_Z^2 = \frac{1}{N-1} \sum\limits_{i \in \Omega} (Z_i - \overline{Z})^2$

$S_{y_1 x_1} = \frac{1}{n_1 - 1} \sum\limits_{i \in S_1} (X_i - \overline{X})(Y_i - \overline{Y})$ $\qquad$ $S_{y_1 z_1} = \frac{1}{n_1 - 1} \sum\limits_{i \in S_1} (Y_i - \overline{Y})(Z_i - \overline{Z})$

$S_{x_1 z_1} = \frac{1}{n_1 - 1} \sum\limits_{i \in S_1} (X_i - \overline{X})(Z_i - \overline{Z})$ $\qquad$ $S_{x_1}^2 = \frac{1}{n_1 - 1} \sum\limits_{i \in S_1} (X_i - \overline{X})^2$

$S_{y_1}^2 = \frac{1}{n_1-1}\sum_{i\in S_1}(Y_i-\overline{Y})^2$ $\qquad$ $S_{z_1}^2 = \frac{1}{n_1-1}\sum_{i\in S_1}(Z_i-\overline{Z})^2$

$\theta_{a,b} = \frac{1}{a}-\frac{1}{b}$ $\qquad$ $k_{YX}=\rho_{YX}\frac{C_Y}{C_X}$ $\qquad$ $k_{YZ}=\rho_{YZ}\frac{C_Y}{C_Z}$ $\qquad$ $k_{XZ}=\rho_{XZ}\frac{C_X}{C_Z}$

3 Materials and Methods

Singh and Vishwakarma (2007) suggested an exponential ratio type estimator for estimating $\overline{Y}$ in double sampling

$$\overline{y}_{Re}^{d} = \overline{y}_n \exp\left(\frac{\overline{x}_1-\overline{x}_n}{\overline{x}_1+\overline{x}_n}\right)$$

Choudhury and Singh (2012) suggested a chain dual to ratio estimator of $\overline{Y}$ in double sampling

$$\overline{y}_{dRe}^{dc} = \overline{y}_n\left\{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_n}{(n_1-n)\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}}\right\}$$

Motivated by Singh and Vishwakarma (2007) and Choudhury and Singh (2012), we suggest a new exponential dual-to-ratio type imputation method in double sampling

$$y_{\cdot i} = \begin{cases} y_i & \text{if } i\in A \\ \frac{\overline{y}_r}{n-r}\Big\{n\exp(\alpha\zeta)-r\Big\} & \text{if } i\in A^c \end{cases}$$

Where

$$\zeta = \left\{\frac{\frac{n_1\overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)-n\overline{x}_r}{(n_1-n)}-\overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)}{\frac{n_1\overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)-n\overline{x}_r}{(n_1-n)}+\overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)}\right\} = \left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)}-\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)}+\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}\Big)}\right\}$$

$$\overline{Z}^* = a\overline{Z}+b \quad \text{and} \quad \overline{z}_1^* = a\overline{z}_1+b$$

The corresponding point estimator of the population mean $\overline{Y}$ is

$$\overline{y}_{GS}^{dc} = \overline{y}_r\exp\left[\alpha\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)}-\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)}+\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}\Big)}\right\}\right]$$

Where α, a and b are constants to be determined such that MSE of the point estimator $\overline{y}_{GS}^{dc}$ is minimum.

Remarks;

a. If $\alpha = 1$, the proposed estimator reduces to the point estimator denoted by t_1

$$t_1 = \overline{y}_r\exp\left[\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)}-\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)}+\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}\Big)}\right\}\right]$$

b. If $b = 0$, the proposed estimator reduces to the estimator denoted by t_2

$$t_2 = \overline{y}_r\exp\left[\alpha\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}}{\overline{z}_1}-n\overline{x}_r}{(n_1-n)}-\overline{x}_1\frac{\overline{Z}}{\overline{z}_1}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}}{\overline{z}_1}-n\overline{x}_r}{(n_1-n)}+\overline{x}_1\frac{\overline{Z}}{\overline{z}_1}\Big)}\right\}\right]$$

c. If $\alpha = 1$ and $b = 0$, the proposed estimator reduces to the estimator denoted by t_3

$$t_3 = \overline{y}_r \exp\left[\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}}{\overline{z}_1}-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\frac{\overline{Z}}{\overline{z}_1}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}}{\overline{z}_1}-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\frac{\overline{Z}}{\overline{z}_1}\Big)}\right\}\right]$$

d. If $\alpha = \rho_{YX}$, the proposed estimator reduces to the estimator denoted by t_4

$$t_4 = \overline{y}_r \exp\left[\rho_{YX}\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}\Big)}\right\}\right]$$

e. If $\alpha = \rho_{YZ}$, the proposed estimator reduces to the estimator denoted by t_5

$$t_5 = \overline{y}_r \exp\left[\rho_{YZ}\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}\Big)}\right\}\right]$$

f. If $\alpha = \rho_{XZ}$, the proposed estimator reduces to the estimator denoted by t_6

$$t_6 = \overline{y}_r \exp\left[\rho_{XZ}\left\{\frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}\Big)}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\frac{\overline{Z}^*}{\overline{z}^*_1}\Big)}\right\}\right]$$

g. If $\alpha = \rho_{YX}$ and $b = 0$, the proposed estimator reduces to the estimator denoted by t_7

$$t_7 = \overline{y}_r \exp\left[\rho_{YX}\left\{\frac{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_r}{(n_1-n)} - \frac{\overline{x}_1}{\overline{z}_1}\overline{Z}\Big)}{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_r}{(n_1-n)} + \frac{\overline{x}_1}{\overline{z}_1}\overline{Z}\Big)}\right\}\right]$$

h. If $\alpha = \rho_{YZ}$ and $b = 0$, the proposed estimator reduces to the estimator denoted by t_8

$$t_8 = \overline{y}_r \exp\left[\rho_{YZ}\left\{\frac{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_r}{(n_1-n)} - \frac{\overline{x}_1}{\overline{z}_1}\overline{Z}\Big)}{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_r}{(n_1-n)} + \frac{\overline{x}_1}{\overline{z}_1}\overline{Z}\Big)}\right\}\right]$$

i. If $\alpha = \rho_{XZ}$ and $b = 0$, the proposed estimator reduces to the estimator denoted by t_9

$$t_9 = \overline{y}_r \exp\left[\rho_{XZ}\left\{\frac{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_r}{(n_1-n)} - \frac{\overline{x}_1}{\overline{z}_1}\overline{Z}\Big)}{\frac{n_1\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-n\overline{x}_r}{(n_1-n)} + \frac{\overline{x}_1}{\overline{z}_1}\overline{Z}\Big)}\right\}\right]$$

j. If $a = \rho_{XZ}$ and $b = C_Z$, the proposed estimator reduces to the estimator denoted by t_{10}

$$t_{10} = \overline{y}_r \exp\left[\alpha\left\{\frac{\frac{n_1\overline{x}_1\left(\frac{\rho_{XZ}\overline{Z}+C_Z}{\rho_{XZ}\overline{z}_1+C_Z}\right)-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\left(\frac{\rho_{XZ}\overline{Z}+C_Z}{\rho_{XZ}\overline{z}_1+C_Z}\right)}{\frac{n_1\overline{x}_1\left(\frac{\rho_{XZ}\overline{Z}+C_Z}{\rho_{XZ}\overline{z}_1+C_Z}\right)-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\left(\frac{\rho_{XZ}\overline{Z}+C_Z}{\rho_{XZ}\overline{z}_1+C_Z}\right)}\right\}\right]$$

4 Properties of the proposed estimator

We have considered the following transformations between the sample mean and the corresponding population mean of the study variables and the auxiliary variables in the sampling of the survey. This transformation shows that the sample mean is an unbiased estimator of the population mean.

For $\delta_i \in (-1,1);\ \forall\ i = 0,1,2,3$

$$\overline{y}_r = (1+\delta_0)\overline{Y} \quad \overline{x}_r = (1+\delta_1)\overline{X} \quad \overline{x}_1 = (1+\delta_2)\overline{X} \quad \overline{z}_1 = (1+\delta_3)\overline{Z} \quad g = \frac{n}{n_1 - n} \quad \phi = \frac{a\overline{Z}}{a\overline{Z}+b}$$

Expected values of δ's are as follows

$$E(\delta_0) = E_1\big[E_2(\delta_0)\big] = E_1\left[E_2\left(\frac{\overline{y}_r - \overline{Y}}{\overline{Y}}\right)\right] = E_1\left[\frac{\overline{y}_1 - \overline{Y}}{\overline{Y}}\right] = \frac{\overline{Y}-\overline{Y}}{\overline{Y}} = 0$$

$$E(\delta_1) = E_1\big[E_2(\delta_1)\big] = E_1\left[E_2\left(\frac{\overline{x}_r - \overline{X}}{\overline{X}}\right)\right] = E_1\left[\frac{\overline{x}_1 - \overline{X}}{\overline{Y}}\right] = \frac{\overline{X}-\overline{X}}{\overline{X}} = 0$$

$$E(\delta_2) = E_1(\delta_2) = E_1\left(\frac{\overline{x}_1 - \overline{X}}{\overline{X}}\right) = \frac{\overline{X}-\overline{X}}{\overline{X}} = 0$$

$$E(\delta_3) = E_1(\delta_3) = E_1\left(\frac{\overline{z}_1 - \overline{Z}}{\overline{Z}}\right) = \frac{\overline{Z}-\overline{Z}}{\overline{Z}} = 0$$

$$\begin{aligned}
E(\delta_0^2) &= E_1\left[E_2\left(\frac{\overline{y}_r - \overline{Y}}{\overline{Y}}\right)^2\right] = \frac{1}{\overline{Y}^2}E_1\big[E_2\{(\overline{y}_r - \overline{y}_1) + (\overline{y}_1 - \overline{Y})\}^2\big] \\
&= \frac{1}{\overline{Y}^2}E_1\left[E_2\left\{(\overline{y}_r - \overline{y}_1)^2 + (\overline{y}_1 - \overline{Y})^2 + 2(\overline{y}_r - \overline{y}_1)(\overline{y_1} - \overline{Y})\right\}\right] \\
&= \frac{1}{\overline{Y}^2}E_1\left[\left(\frac{1}{r} - \frac{1}{n_1}\right)s_{y_1}^2 + (\overline{y_1} - \overline{Y})^2 + 0\right] \\
&= \frac{1}{\overline{Y}^2}\left[\left(\frac{1}{r} - \frac{1}{n_1}\right)S_Y^2 + \left(\frac{1}{n_1} - \frac{1}{N}\right)S_Y^2\right] \\
&= \left(\frac{1}{r} - \frac{1}{N}\right)C_Y^2
\end{aligned}$$

$$\begin{aligned}
E(\delta_1^2) &= E_1\left[E_2\left(\frac{\overline{x}_r - \overline{X}}{\overline{X}}\right)^2\right] = \frac{1}{\overline{X}^2}E_1\left[E_2\left\{(\overline{x}_r - \overline{x}_1) + (\overline{x}_1 - \overline{X})\right\}^2\right] \\
&= \frac{1}{\overline{X}^2}E_1\left[E_2\Big\{(\overline{x}_r - \overline{x}_1)^2 + (\overline{x}_1 - \overline{X})^2 + 2(\overline{x}_r - \overline{x}_1)(\overline{x}_1 - \overline{X})\Big\}\right] \\
&= \frac{1}{\overline{X}^2}E_1\left[\left(\frac{1}{r} - \frac{1}{n_1}\right)s_{x_1}^2 + (\overline{x}_1 - \overline{X})^2 + 0\right] \\
&= \frac{1}{\overline{X}^2}\left[\left(\frac{1}{r} - \frac{1}{n_1}\right)S_X^2 + \left(\frac{1}{n_1} - \frac{1}{N}\right)S_X^2\right] \\
&= \left(\frac{1}{r} - \frac{1}{N}\right)C_X^2
\end{aligned}$$

$$E(\delta_2^2) = E_1\left(\frac{\overline{x}_1 - \overline{X}}{\overline{X}}\right)^2 = \frac{1}{\overline{X}^2}E_1(\overline{x}_1 - \overline{X})^2 = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_X^2$$

$$E(\delta_3^2) = E_1\left(\frac{\overline{z}_1 - \overline{Z}}{\overline{Z}}\right)^2 = \frac{1}{\overline{Z}^2}E_1(\overline{z}_1 - \overline{Z})^2 = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_Z^2$$

$$\begin{aligned}
E(\delta_0\delta_1) &= \frac{1}{\overline{X}\overline{Y}}E_1\Big[E_2(\overline{y}_r - \overline{Y})(\overline{x}_r - \overline{X})\Big] \\
&= \frac{1}{\overline{Y}\overline{Z}}E_1\Big[E_2\Big\{(\overline{y}_r - \overline{y}_1 + \overline{y}_1 - \overline{Y})(\overline{x}_r - \overline{x}_1 + \overline{x}_1 - \overline{X})\Big\}\Big] \\
&= \frac{1}{\overline{X}\overline{Y}}E_1\Big[E_2\Big\{(\overline{y}_r - \overline{y}_1)(\overline{x}_r - \overline{x}_1) + (\overline{y}_1 - \overline{Y})(\overline{x}_1 - \overline{X}) + 0\Big\}\Big] \\
&= \frac{1}{\overline{X}\overline{Y}}\left[\left(\frac{1}{r} - \frac{1}{n_1}\right)s_{y_1x_1} + (\overline{y}_1 - \overline{Y})(\overline{x}_1 - \overline{X})\right] \\
&= \frac{1}{\overline{X}\overline{Y}}\left[\left(\frac{1}{r} - \frac{1}{n_1}\right)S_{YX} + \left(\frac{1}{n_1} - \frac{1}{N}\right)S_{YX}\right] \\
&= \left(\frac{1}{r} - \frac{1}{N}\right)\rho_{YX}C_YC_X
\end{aligned}$$

$$\begin{aligned}
E(\delta_0\delta_2) &= \frac{1}{\overline{Y}\overline{X}}E_1\Big[E_2(\overline{y}_r - \overline{Y})(\overline{x}_1 - \overline{X})\Big] \\
&= \frac{1}{\overline{Y}\overline{X}}E_1\left[\Big\{(\overline{x}_1 - \overline{X})E_2(\overline{y}_r - \overline{Y})\Big\}\right] \\
&= \frac{1}{\overline{Y}\overline{X}}E_1\left[\Big\{(\overline{x}_1 - \overline{X})(\overline{y}_1 - \overline{Y})\Big\}\right] \\
&= \frac{1}{\overline{Y}\overline{X}}\left(\frac{1}{n_1} - \frac{1}{N}\right)S_{YX} \\
&= \left(\frac{1}{n_1} - \frac{1}{N}\right)\rho_{YX}C_YC_X
\end{aligned}$$

$$\begin{aligned}
E(\delta_0\delta_3) &= \frac{1}{\overline{Y}\overline{Z}}E_1\Big[E_2(\overline{y}_r - \overline{Y})(\overline{z}_1 - \overline{Z})\Big] \\
&= \frac{1}{\overline{Y}\overline{Z}}E_1\Big[\Big\{(\overline{z}_1 - \overline{Z})E_2(\overline{y}_r - \overline{Y})\Big\}\Big] \\
&= \frac{1}{\overline{Y}\overline{Z}}E_1\left[\Big\{(\overline{z}_1 - \overline{Z})(\overline{y}_1 - \overline{Y})\Big\}\right] \\
&= \frac{1}{\overline{Y}\overline{Z}}\left(\frac{1}{n_1} - \frac{1}{N}\right)S_{YZ} \\
&= \left(\frac{1}{n_1} - \frac{1}{N}\right)\rho_{YZ}C_YC_Z
\end{aligned}$$

$$\begin{aligned}
E(\delta_1\delta_2) &= \frac{1}{\overline{X}\overline{X}}E_1\Big[E_2(\overline{x}_r - \overline{X})(\overline{x}_1 - \overline{X})\Big] \\
&= \frac{1}{\overline{X}^2}E_1\Big[\Big\{(\overline{x}_1 - \overline{X})E_2(\overline{x}_r - \overline{X})\Big\}\Big] \\
&= \frac{1}{\overline{X}^2}E_1\Big[\Big\{(\overline{x}_1 - \overline{X})(\overline{x}_1 - \overline{X})\Big\}\Big] \\
&= \frac{1}{\overline{X}^2}\left(\frac{1}{n_1} - \frac{1}{N}\right)S_X^2 \\
&= \left(\frac{1}{n_1} - \frac{1}{N}\right)C_X^2
\end{aligned}$$

$$\begin{aligned}E(\delta_1\delta_3) &= \frac{1}{\overline{X}\overline{Z}}E_1\Big[E_2(\overline{x}_r-\overline{X})(\overline{z}_1-\overline{Z})\Big]\\ &= \frac{1}{\overline{X}\overline{Z}}E_1\left[\Big\{(\overline{z}_1-\overline{Z})E_2(\overline{x}_r-\overline{X})\Big\}\right]\\ &= \frac{1}{\overline{Y}\overline{Z}}E_1\left[\Big\{(\overline{z}_1-\overline{Z})(\overline{x}_1-\overline{X})\Big\}\right]\\ &= \frac{1}{\overline{X}\overline{Z}}\left(\frac{1}{n_1}-\frac{1}{N}\right)S_{XZ}\\ &= \left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{XZ}C_XC_Z\end{aligned}$$

$$\begin{aligned}E(\delta_2\delta_3) &= \frac{1}{\overline{Z}\overline{X}}E_1\Big[E_2(\overline{z}_1-\overline{Z})(\overline{x}_1-\overline{X})\Big]\\ &= \frac{1}{\overline{Z}\overline{X}}E_1\Big[(\overline{x}_1-\overline{X})(\overline{z}_1-\overline{Z})\Big]\\ &= \frac{1}{\overline{Z}\overline{X}}\left(\frac{1}{n_1}-\frac{1}{N}\right)S_{XZ}\\ &= \left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{XZ}C_XC_Z\end{aligned}$$

Thus, we have

$$\begin{array}{llll}E(\delta_0)=0 & E(\delta_1)=0 & E(\delta_2)=0 & E(\delta_3)=0\\ E(\delta_0^2)=\theta_{r,N}C_Y^2 & E(\delta_1^2)=\theta_{r,N}C_X^2 & E(\delta_2^2)=E(\delta_1\delta_2)=\theta_{n_1,N}C_X^2 & \\ E(\delta_3^2)=\theta_{n_1,N}C_Z^2 & E(\delta_0\delta_1)=\theta_{r,N}\rho_{YX}C_YC_X & E(\delta_0\delta_2)=\theta_{n_1,N}\rho_{YX}C_YC_X & \\ E(\delta_0\delta_3)=\theta_{n_1,N}\rho_{YZ}C_YC_Z & E(\delta_1\delta_3)=E(\delta_2\delta_3)=\theta_{n_1,N}\rho_{XZ}C_XC_Z & & \end{array}$$

Expressing $\overline{y}_{GS}^{dc}$ in terms of δ 's up to $O(n^{-1})$, we have

$$\begin{aligned}\overline{y}_{GS}^{dc} &= \overline{Y}(1+\delta_0)\exp\left[\alpha\left\{\frac{(1+g)(1+\delta_2)(1+\phi\delta_3)^{-1}-g(1+\delta_1)-(1+\delta_2)(1+\phi\delta_3)^{-1}}{(1+g)(1+\delta_2)(1+\phi\delta_3)^{-1}-g(1+\delta_1)+(1+\delta_2)(1+\phi\delta_3)^{-1}}\right\}\right]\\ &= \overline{Y}(1+\delta_0)\exp\left[\alpha\left\{1-\frac{2(1+\delta_2)(1+\phi\delta_3)^{-1}}{(1+g)(1+\delta_2)(1+\phi\delta_3)^{-1}-g(1+\delta_1)+(1+\delta_2)(1+\phi\delta_3)^{-1}}\right\}\right]\\ &= \overline{Y}(1+\delta_0)\exp\left[\alpha\left\{1-\frac{2(1+\delta_2)}{(1+g)(1+\delta_2)-g(1+\delta_1)(1+\phi\delta_3)+(1+\delta_2)}\right\}\right]\\ &= \overline{Y}(1+\delta_0)\exp\left[\alpha\left\{1-\frac{(1+\delta_2)}{1+\delta_2+\frac{g}{2}(\delta_2-\delta_1-\phi\delta_3-\phi\delta_1\delta_3}\right\}\right]\\ &= \overline{Y}(1+\delta_0)\exp\left[\alpha\left\{1-(1+\delta_2)\Big\{(1+\delta_2+\frac{g}{2}(\delta_2-\delta_1-\phi\delta_3-\phi\delta_1\delta_3)\Big\}^{-1}\right\}\right]\\ &= \overline{Y}(1+\delta_0)\exp\Bigg[\frac{g}{2}\left(\delta_2-\delta_1-\phi\delta_3-\phi\delta_1\delta_3-\delta_2^2+\delta_1\delta_2+\phi\delta_2\delta_3\right)\\ &\qquad -\frac{g^2}{4}\left(\delta_2^2+\delta_1^2+\phi^2\delta_3^2-2\delta_1\delta_2-2\phi\delta_2\delta_3+2\phi\delta_1\delta_3\right)\Bigg]\\ &= \overline{Y}(1+\delta_0)\exp\left\{\alpha\left(\frac{g}{2}P-\frac{g^2}{4}Q\right)\right\}\\ &= \overline{Y}(1+\delta_0)\left\{1+\alpha\left(\frac{g}{2}P-\frac{g^2}{4}Q\right)+\frac{\alpha^2}{2}\left(\frac{g}{2}P-\frac{g^2}{4}Q\right)^2\right\}\end{aligned}$$

$$
\begin{aligned}
&= \overline{Y}\left\{1 + \delta_0 + \alpha\left(\frac{g}{2}P - \frac{g^2}{4}Q\right) + \frac{\alpha^2 g^2}{8}Q + \frac{\alpha g}{2}(\delta_0\delta_2 - \delta_0\delta_1 - \phi\delta_0\delta_3\right\}\\
&= \overline{Y}\left\{1 + \delta_0 + \frac{\alpha g}{2}\left(P + \delta_0\delta_2 - \delta_0\delta_1 - \phi\delta_0\delta_3\right) - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2 Q\right\}\\
&= \overline{Y}\left\{1 + \delta_0 + \frac{\alpha g}{2}P^* - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2 Q\right\}
\end{aligned}
$$

Thus

$$
\overline{y}_{GS}^{dc} - \overline{Y} = \overline{Y}\left\{\delta_0 + \frac{\alpha g}{2}P^* - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2 Q\right\} \tag{1}
$$

Where

$$
\begin{aligned}
P &= \delta_2 - \delta_1 - \phi\delta_3 - \phi\delta_1\delta_3 - \delta_2^2 + \delta_1\delta_2 + \phi\delta_2\delta_3\\
Q &= \delta_2^2 + \delta_1^2 + \phi^2\delta_3^2 - 2\delta_1\delta_2 - 2\phi\delta_2\delta_3 + 2\phi\delta_1\delta_3\\
P^* &= P + \delta_0\delta_2 - \delta_0\delta_1 - \phi\delta_0\delta_3
\end{aligned}
$$

Theorem 4.1: $Bias$ and MSE of the proposed estimator up to $O(n^{-1})$ are

$$
B(\overline{y}_{GS}^{dc}) = -\overline{Y}\left\{\left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi^2 C_Z^2\Big) + \frac{\alpha g}{2}\Big(\theta_{r,n_1}\rho_{YX}C_Y C_X + \theta_{n_1,N}\phi\rho_{YZ}C_Y C_Z\Big)\right\} \tag{2}
$$

$$
M(\overline{y}_{GS}^{dc}) = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 + \frac{\alpha^2 g^2}{4}\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi^2 C_Z^2\Big) - \alpha g\Big(\theta_{r,n_1}\rho_{YX}C_Y C_X + \theta_{n_1,N}\phi\rho_{YZ}C_Y C_Z\Big)\right\} \tag{3}
$$

Proof: Taking expectations on both sides of equation (1) and putting the corresponding expected values, we have

$$
\begin{aligned}
B(\overline{y}_{GS}^{dc}) &= \overline{Y}\left\{E(\delta_0) + \frac{\alpha g}{2}E(P^*) - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2 E(Q)\right\}\\
&= \overline{Y}\Bigg[\frac{\alpha g}{2}\Big\{E(\delta_0\delta_2) - E(\delta_0\delta_1) - \phi E(\delta_0\delta_3)\Big\} - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2\Big\{E(\delta_1^2)\\
&\qquad + E(\delta_2^2) + \phi^2 E(\delta_3^2) - 2E(\delta_1\delta_2)\Big\}\Bigg]\\
&= \overline{Y}\left[\frac{\alpha g}{2}\Big\{-\theta_{r,n_1}\rho_{YX}C_Y C_X - \phi\theta_{n_1,N}\rho_{YZ}C_Y C_Z\Big\} - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2\Big\{\theta_{r,n_1}C_X^2 + \phi^2\theta_{n_1,N}C_Z^2\Big\}\right]\\
&= -\overline{Y}\left[\left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2\Big\{\theta_{r,n_1}C_X^2 + \phi^2\theta_{n_1,N}C_Z^2\Big\} + \frac{\alpha g}{2}\Big\{\theta_{r,n_1}\rho_{YX}C_Y C_X + \phi\theta_{n_1,N}\rho_{YZ}C_Y C_Z\Big\}\right]
\end{aligned}
$$

Squaring both the sides of equation (1) and taking expectation up to $O(n^{-1})$, we have

$$
\begin{aligned}
E(\overline{y}_{GS}^{dc} - \overline{Y})^2 &= \overline{Y}^2 E\left[\left\{\delta_0 + \frac{\alpha g}{2}P^* - \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)g^2 Q\right\}^2\right]\\
&= \overline{Y}^2\left\{E(\delta_0^2) + \frac{\alpha^2 g^2}{4}E(P_1) + \alpha g E(Q_1)\right\}
\end{aligned} \tag{4}
$$

Where

$$P_1 = \delta_1^2 + \delta_2^2 + \phi^2\delta_3^2 - 2\left(\delta_1\delta_2 + \phi\delta_2\delta_3 - \phi\delta_1\delta_3\right)$$
$$Q_1 = \delta_0\delta_2 - \delta_0\delta_1 - \phi\delta_0\delta_3$$

Putting the corresponding expected values, we have

$$\begin{aligned} E(P_1) &= \theta_{r,N}C_X^2 + \theta_{n_1,N}C_X^2 + \phi^2\theta_{n_1,N}C_Z^2 - 2\theta_{n_1,N}C_X^2 - 2\phi\left(\theta_{n_1,N}\rho_{XZ}C_XC_Z - \theta_{n_1,N}\rho_{XZ}C_XC_Z\right) \\ &= \theta_{r,N}C_X^2 - \theta_{n_1,N}C_X^2 + \phi^2\theta_{n_1,N}C_Z^2 \\ &= \theta_{r,n_1}C_X^2 + \phi^2\theta_{n_1,N}C_Z^2 \end{aligned}$$

And

$$\begin{aligned} E(Q_1) &= \theta_{n_1,N}\rho_{YX}C_YC_X - \theta_{r,N}\rho_{YX}C_YC_X - \phi\theta_{n_1,N}\rho_{YZ}C_YC_Z \\ &= -\theta_{r,n_1}\rho_{YX}C_YC_X - \phi\theta_{n_1,N}\rho_{YZ}C_YC_Z \end{aligned}$$

Thus

$$M(\overline{y}_{GS}^{dc}) = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 + \frac{\alpha^2g^2}{4}\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi^2C_Z^2\Big) - \alpha g\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi\rho_{YZ}C_YC_Z\Big)\right\}$$

Theorem 4.2: MSE of the proposed estimator $\overline{y}_{GS}^{dc}$ at optimum values of α and ϕ is

$$M(\overline{y}_{GS}^{dc})_{min} = \overline{Y}^2\Big\{\left(\theta_{r,N} - \theta_{r,n_1}\rho_{YX}^2\right)C_Y^2 - \theta_{n_1,N}\rho_{YZ}^2C_Y^2\Big\}$$

Proof: Differentiating equation (3) with respect to ϕ and α and equating to zero, we have

$$\alpha g\phi = 2\rho_{YZ}\frac{C_Y}{C_Z} \quad \text{and} \quad \alpha g\left(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi^2C_Z^2\right) = 2\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi\rho_{YZ}C_YC_Z\right)$$

Solving the above two equations, we have

$$\alpha_{opt} = \frac{2}{g}\rho_{YX}\frac{C_Y}{C_X} \quad \text{and} \quad \phi_{opt} = \frac{\rho_{YZ}}{\rho_{YX}}\frac{C_X}{C_Z}$$

Again at $\alpha = \alpha_{opt}$ and $\phi = \phi_{(opt)}$, we have

$$\left[\frac{\delta^2}{\delta\alpha^2}M(\overline{y}_{GS}^{dc})\right] = \overline{Y}^2\frac{g^2}{2}(\theta_{r,N}C_X^2 + \phi^2\theta_{n_1,N}C_Z^2)C_X^2 > 0$$

$$\left[\frac{\delta^2}{\delta\phi^2}M(\overline{y}_{GS}^{dc})\right] = \overline{Y}^2\frac{\alpha^2g^2}{2}\left(\theta_{n_1,N}C_Z^2\right) > 0 \qquad \text{and}$$

$$\left[\frac{\delta^2}{\delta\alpha^2}M(\overline{y}_{GS}^{dc})\frac{\delta^2}{\delta\phi^2}M(\overline{y}_{GS}^{dc}) - \left\{\frac{\delta^2}{\delta\alpha\delta\phi}M(\overline{y}_{GS}^{dc})\right\}^2\right] = \overline{Y}^4\theta_{r,n_1}\theta_{n_1,N}\Big(g\rho_{YX}C_YC_Z\Big)^2 > 0$$

Thus, putting the value of $\alpha = \alpha_{opt}$ and $\phi = \phi_{opt}$, we have minimum Mean Square Error of $\overline{y}_{GS}^{dc}$

$$M(\overline{y}_{GS}^{dc})_{min} = \overline{Y}^2\Big\{\left(\theta_{r,N} - \theta_{r,n_1}\rho_{YX}^2\right)C_Y^2 - \theta_{n_1,N}\rho_{YZ}^2C_Y^2\Big\}$$

Theorem 4.3: Minimum $Bias$ and MSE of the estimator t_1 are

$$B(t_1)_{min} = -\overline{Y}\left\{\theta_{r,n_1}\left(\frac{g^2}{8}C_X^2 + \frac{g}{2}\rho_{YX}C_YC_X\right) + \frac{3}{2}\theta_{n_1,N}\rho_{YZ}^2C_Y^2\right\}$$
$$M(t_1)_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \theta_{n_1,N}\left(g\rho_{YX}C_YC_X - \frac{g^2}{4}C_X^2\right) - \theta_{n_1,N}\rho_{YZ}^2C_Y^2\right\}$$

Theorem 4.4: Minimum $Bias$ and MSE of the estimator t_2 are

$$B(t_2)_{min} = -\overline{Y}\left\{\frac{g}{2}\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right) + \frac{1}{2}\frac{\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right)^2}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2}\right\}$$

$$M(t_2)_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \frac{\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right)^2}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2}\right\}$$

Theorem 4.5: Minimum $Bias$ and MSE of the estimator t_3 are

$$B(t_3)_{min} = -\overline{Y}\left\{\frac{1}{8}g^2\left(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\right) + \frac{g}{2}\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right)\right\}$$

$$M(t_3)_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - g\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right) + \frac{g^2}{4}\left(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\right)\right\}$$

Theorem 4.6: Minimum $Bias$ and MSE of the estimator t_4 are

$$B(t_4)_{min} = -\overline{Y}\left\{\left(\frac{\rho_{YX}}{4} - \frac{\rho_{YX}^2}{8}\right)g^2\left(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi_{4(opt)}^2C_Z^2\right) + \frac{\rho_{YX}}{2}g\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi_{4(opt)}\rho_{YZ}C_YC_Z\right)\right\}$$

$$M(t_4)_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \theta_{r,n_1}\left(g\rho_{YX}^2C_YC_X - \frac{g^2}{4}\rho_{YX}^2C_X^2\right) - \theta_{n_1,N}\rho_{YZ}^2C_Y^2\right\}$$

Where $\phi_{4(opt)} = \frac{2}{g}\frac{\rho_{YZ}C_Y}{\rho_{YX}C_Z}$

Theorem 4.7: Minimum $Bias$ and MSE of the estimator t_5 are

$$B(t_5)_{min} = -\overline{Y}\left\{\left(\frac{\rho_{YZ}}{4} - \frac{\rho_{YZ}^2}{8}\right)g^2\left(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi_{5(opt)}^2C_Z^2\right) + \frac{\rho_{YX}}{2}g\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi_{5(opt)}\rho_{YZ}C_YC_Z\right)\right\}$$

$$M(t_5)_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \theta_{r,n_1}\left(g\rho_{YX}\rho_{YZ}C_YC_X - \frac{g^2}{4}\rho_{YZ}^2C_X^2\right) - \theta_{n_1,N}\rho_{YZ}^2C_Y^2\right\}$$

Where $\phi_{5(opt)} = \frac{2}{g}\frac{C_Y}{C_Z}$

Theorem 4.8: Minimum $Bias$ and MSE of the estimator t_6 are

$$B(t_6)_{min} = -\overline{Y}\left\{\left(\frac{\rho_{XZ}}{4} - \frac{\rho_{XZ}^2}{8}\right)g^2\left(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi_{6(opt)}^2C_Z^2\right) + \frac{\rho_{YX}}{2}g\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi_{6(opt)}\rho_{YZ}C_YC_Z\right)\right\}$$

$$M(t_6)_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \theta_{r,n_1}\left(g\rho_{XZ}\rho_{YX}C_YC_X - \frac{g^2}{4}\rho_{XZ}^2C_X^2\right) - \theta_{n_1,N}\rho_{YZ}^2C_Y^2\right\}$$

Where $\phi_{6(opt)} = \frac{2}{g}\frac{\rho_{YZ}C_Y}{\rho_{XZ}C_Z}$

Theorem 4.9: Minimum $Bias$ and MSE of the estimator t_7 are

$$B(t_7)_{min} = -\overline{Y}\Bigg\{ \left(\frac{\rho_{YX}}{4} - \frac{\rho_{YX}^2}{8}\right) g^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\Big) + \frac{\rho_{YX}}{2}g\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\Big)\Bigg\}$$

$$M(t_7)_{min} = \overline{Y}^2\Bigg\{\theta_{r,N}C_Y^2 - g\rho_{YX}\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\Big) + \frac{g^2}{4}\rho_{YX}^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\Big)\Bigg\}$$

Theorem 4.10: Minimum $Bias$ and MSE of the estimator t_8 are

$$B(t_8)_{min} = -\overline{Y}\Bigg\{ \left(\frac{\rho_{YZ}}{4} - \frac{\rho_{YZ}^2}{8}\right) g^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\Big) + \frac{\rho_{YZ}}{2}g\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\Big)\Bigg\}$$

$$M(t_8)_{min} = \overline{Y}^2\Bigg\{\theta_{r,N}C_Y^2 - g\rho_{YZ}\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\Big) + \frac{g^2}{4}\rho_{YZ}^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\Big)\Bigg\}$$

Theorem 4.11: Minimum $Bias$ andMSE of the estimator t_9 are

$$B(t_9)_{min} = -\overline{Y}\Bigg\{ \left(\frac{\rho_{XZ}}{4} - \frac{\rho_{XZ}^2}{8}\right) g^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\Big) + \frac{\rho_{XZ}}{2}g\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\Big)\Bigg\}$$

$$M(t_9)_{min} = \overline{Y}^2\Bigg\{\theta_{r,N}C_Y^2 - g\rho_{XZ}\Big(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\Big) + \frac{g^2}{4}\rho_{XZ}^2\Big(\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2\Big)\Bigg\}$$

Theorem 4.12: Minimum $Bias$ and MSE of the estimator t_{10} are

$$B(t_{10})_{min} = -\overline{Y}\Bigg\{\frac{g}{2}\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi_0\rho_{YZ}C_YC_Z\right) + \frac{1}{2}\frac{\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi_0\rho_{YZ}C_YC_Z\right)^2}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi_0^2C_Z^2}\Bigg\}$$

$$M(t_{10})_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \frac{\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\phi_0\rho_{YZ}C_YC_Z\right)^2}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}\phi_0^2C_Z^2}\right\}$$

Where $$\phi_0 = \frac{\overline{Z}}{\overline{Z} + (C_Z/\rho_{XZ})}$$

5 Some Existing Imputation Methods

Here we consider some existing imputation methods for estimating the population mean of the study variable $\overline{Y}$ where the population mean of the auxiliary variable $\overline{X}$ is not used

5.1 $Mean\ Method\ of\ Imputation$

Under this imputation method

$$y_{\cdot i} = \begin{cases} y_i & \text{if } i \in A \\ \overline{y}_r & \text{if } i \in A^c \end{cases}$$

The corresponding point estimator is

$$\overline{y}_m = \overline{y}_r$$

Lemma 1: The Bias and Variance of the point estimator $\overline{y}_m$ are

$$B(\overline{y}_m) = 0$$
$$Var(\overline{y}_m) = \overline{Y}^2 \theta_{r,N} C_Y^2$$

5.2 $Ratio\ Method\ of\ Imputation$

Under this imputation method

$$y_{\cdot i} = \begin{cases} y_i & \text{if } i \in A \\ \hat{b} x_i & \text{if } i \in A^c \end{cases}$$

Where $$\hat{b} = \frac{\sum\limits_{i \in R} y_i}{\sum\limits_{i \in R} x_i}$$

The corresponding point estimator is

$$\overline{y}_R = \overline{y}_r \frac{\overline{x}_n}{\overline{x}_r}$$

Lemma 2: The $Bias$ and MSE of the point estimator $\overline{y}_R$ are

$$B(\overline{y}_R) = \theta_{r,N} \overline{Y} \Big(1 - k_{YX}\Big) C_X^2$$
$$M(\overline{y}_R) = \overline{Y}^2 \Big\{ \theta_{r,N} C_Y^2 + \theta_{r,n} (1 - 2k_{YX}) C_X^2 \Big\}$$

5.3 $Compromised\ Method\ of\ Imputation\ suggested\ by\ Singh\ and\ Horn$ (2000)

Under this imputation method

$$y_{\cdot i} = \begin{cases} \beta \frac{n}{r} y_i + (1-\beta) \hat{b} x_i & \text{if } i \in A \\ (1-\beta) \hat{b} x_i & \text{if } i \in A^c \end{cases}$$

The corresponding point estimator is

$$\overline{y}_{Comp} = \beta \overline{y}_r + (1-\beta) \overline{y}_r \frac{\overline{x}_n}{\overline{x}_r}$$

Where β is a suitable chosen constant to be determined such that MSE of $\overline{y}_{Comp}$ is minimum.

Lemma 3: The $Bias$ and MSE of the point estimator $\overline{y}_{Comp}$ at $\beta_{opt} = 1 - \rho_{YX}\frac{C_Y}{C_X}$ are

$$B(\overline{y}_{Comp})_{min} = \overline{Y}\theta_{r,n}\Big(C_X - \rho_{YX}C_Y\Big)\rho_{YX}C_Y$$

$$M(\overline{y}_{Comp})_{min} = \overline{Y}^2\Big\{\theta_{r,N}C_Y^2 - \theta_{r,n}\rho_{YX}^2C_Y^2\Big\}$$

5.4 *Ratio cum Product Compromised Method of Imputation in Double Sampling suggested by Nath and Singh* (2018)

Under this imputation methods

$$y_{1i} = \begin{cases} \alpha_1\frac{n}{r}y_i\frac{\overline{x}_1}{\overline{x}_n} + (1-\alpha_1)\overline{y}_r\frac{\overline{x}_n}{\overline{x}_1} & \text{if } i \in A \\ (1-\alpha_1)\overline{y}_r\frac{\overline{x}_n}{\overline{x}_1} & \text{if } i \in A^c \end{cases}$$

$$y_{2i} = \begin{cases} \alpha_1\frac{n}{r}y_i\frac{\overline{x}_n}{\overline{x}_r} + (1-\alpha_1)\overline{y}_r\frac{\overline{x}_r}{\overline{x}_n} & \text{if } i \in A \\ (1-\alpha_1)\overline{y}_r\frac{\overline{x}_r}{\overline{x}_n} & \text{if } i \in A^c \end{cases}$$

$$y_{3i} = \begin{cases} \alpha_1\frac{n}{r}y_i\frac{\overline{x}_1}{\overline{x}_r} + (1-\alpha_1)\overline{y}_r\frac{\overline{x}_r}{\overline{x}_1} & \text{if } i \in A \\ (1-\alpha_1)\overline{y}_r\frac{\overline{x}_r}{\overline{x}_1} & \text{if } i \in A^c \end{cases}$$

Where α_1, α_2 and α_3 are suitably chosen constants such that the MSE of the resulting point estimator is minimum. The corresponding point estimators are

$$\overline{y}_{IRP1}^d = \overline{y}_r\left\{\alpha_1\frac{\overline{x}_1}{\overline{x}_n} + (1-\alpha_1)\frac{\overline{x}_n}{\overline{x}_1}\right\}$$

$$\overline{y}_{IRP2}^d = \overline{y}_r\left\{\alpha_1\frac{\overline{x}_n}{\overline{x}_r} + (1-\alpha_1)\frac{\overline{x}_r}{\overline{x}_n}\right\}$$

$$\overline{y}_{IRP3}^d = \overline{y}_r\left\{\alpha_1\frac{\overline{x}_1}{\overline{x}_r} + (1-\alpha_1)\frac{\overline{x}_r}{\overline{x}_1}\right\}$$

Lemma 4: The minimum $Bias$ and MSE of the point estimators $\overline{y}_{IRP1}^d.\overline{y}_{IRP2}^d$ and $\overline{y}_{IRP3}^d$ are

$$B(\overline{y}_{IRP1}^d)_{min} = \overline{Y}\left\{\frac{1}{2}\theta_{n,n_1}\left(C_X^2 + \rho_{YX}C_YC_X\right) - \theta_{n,n_1}\rho_{YX}^2C_Y^2\right\}$$

$$B(\overline{y}_{IRP2}^d)_{min} = \overline{Y}\left\{\frac{1}{2}\theta_{r,n}\left(C_X^2 + \rho_{YX}C_YC_X\right) - \theta_{r,n}\rho_{YX}^2C_Y^2\right\}$$

$$B(\overline{y}_{IRP3}^d)_{min} = \overline{Y}\left\{\frac{1}{2}\theta_{r,n_1}\left(C_X^2 + \rho_{YX}C_YC_X\right) - \theta_{r,n_1}\rho_{YX}^2C_Y^2\right\}$$

$$M(\overline{y}_{IRP1}^d)_{min} = \overline{Y}^2\Big\{\theta_{r,N}C_Y^2 - \theta_{n,n_1}\rho_{YX}^2C_Y^2\Big\}$$

$$M(\overline{y}_{IRP2}^d)_{min} = \overline{Y}^2\Big\{\theta_{r,N}C_Y^2 - \theta_{r,n}\rho_{YX}^2C_Y^2\Big\}$$

$$M(\overline{y}_{IRP3}^d)_{min} = \overline{Y}^2\Big\{\theta_{r,N}C_Y^2 - \theta_{r,n_1}\rho_{YX}^2C_Y^2\Big\}$$

5.5 *Exponential chain Ratio Method of imputation in Double Sampling suggested by Kumar et al.*(2017)

Under this imputation method

$$y_{\cdot i} = \begin{cases} y_i & \text{if } i \in A \\ \frac{1}{n-r}\Big\{n\overline{y}_r\psi - r\overline{y}_r\Big\} & \text{if } i \in A^c \end{cases}$$

Where $\psi = \exp\left\{\alpha\left(\frac{\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-\overline{x}_r}{\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}+\overline{x}_r}\right)\right\}$

Here α is a suitable chosen constant such that MSE of the resulting point estimator is minimum. The corresponding point estimator is

$$\overline{y}_{IRe}^{dc} = \overline{y}_r \exp\left\{\alpha\left(\frac{\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}-\overline{x}_r}{\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}+\overline{x}_r}\right)\right\}$$

Lemma 5: Minimum $Bias$ and MSE of the point estimator $\overline{y}_{IRe}^{dc}$ are

$$B(\overline{y}_{IRe}^{dc})_{min} = \overline{Y}\left\{\frac{1}{2}\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right) - \frac{1}{2}\frac{\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right)^2}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2}\right\}$$

$$M(\overline{y}_{IRe}^{dc})_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 - \frac{\left(\theta_{r,n_1}\rho_{YX}C_YC_X + \theta_{n_1,N}\rho_{YZ}C_YC_Z\right)^2}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2}\right\}$$

5.6 *Factor Type chain Ratio Method of Imputations in Double Sampling suggested by Thakur and Shukla*(2022)

Under these imputation methods

$$y_{1i} = \begin{cases} y_i & \text{if } i \in A \\ \frac{1}{n-r}\Big[n\psi_1(k) - r\overline{y}_r\Big] & \text{if } i \in A^c \end{cases}$$

$$y_{2i} = \begin{cases} y_i & \text{if } i \in A \\ \frac{1}{n-r}\Big[n\psi_2(k) - r\overline{y}_r\Big] & \text{if } i \in A^c \end{cases}$$

$$y_{3i} = \begin{cases} y_i & \text{if } i \in A \\ \frac{1}{n-r}\Big[n\psi_3(k) - r\overline{y}_r\Big] & \text{if } i \in A^c \end{cases}$$

The corresponding point estimators are

$$\psi_1(k) = \overline{y}_r\frac{\overline{x}_1}{\overline{x}_r}\left\{\frac{(A+C)\,\overline{Z} + fB\overline{z}_1}{(A+fB)\,\overline{Z} + C\overline{z}_1}\right\}$$

$$\psi_2(k) = \overline{y}_r\frac{\overline{x}_1}{\overline{x}_r}\left\{\frac{(A+C)\,\overline{z}_1 + fB\overline{z}_r}{(A+fB)\,\overline{z}_1 + C\overline{z}_r}\right\}$$

$$\psi_3(k) = \overline{y}_r\frac{\overline{x}_1}{\overline{x}_r}\left\{\frac{(A+C)\,\overline{Z} + fB\overline{z}_r}{(A+fB)\,\overline{Z} + C\overline{z}_r}\right\}$$

$$A = (k-1)(k-2); \quad B = (k-1)(k-4); \quad C = (k-2)(k-3)(k-4); \quad 0 < k < \infty$$

Where k is a constant to be determined, such that the corresponding point estimators has minimum MSE.

Lemma 6: Minimum $Bias$ and MSE of the point estimators $\psi_1(k)$,$\psi_2(k)$ and $\psi_3(k)$ are

$$B[\psi_1(k)]_{min} = \overline{Y}\left\{\theta_{r,N}C_X^2\left(1-k_{YX}\right) - \phi_{opt}\theta_{n_1,N}C_Z^2\left(\phi_{2(opt)} - k_{YZ}\right)\right\}; \qquad \phi_{opt} = -k_{YZ}$$

$$B[\psi_2(k)]_{min} = \overline{Y}\left\{\theta_{r,n_1}C_X^2\left(1-k_{YX}\right) - \phi_{opt}\theta_{r,n_1}C_Z^2\left(\phi_{2(opt)} - k_{YZ} + k_{XZ}\right)\right\}; \quad \phi_{opt} = k_{XZ} - k_{YZ}$$

$$B[\psi_3(k)]_{min} = \overline{Y}\left\{\theta_{r,n_1}C_X^2\left(1-k_{YX}\right) + \phi_{opt}C_Z^2\left(\theta_{r,N}k_{YZ} - \theta_{r,n_1}k_{XZ} - \theta_{r,N}\phi_{2(opt)}\right)\right\}$$

$$\phi_{opt} = \frac{\theta_{r,n_1}k_{XZ} - \theta_{r,N}k_{YZ}}{\theta_{r,N}}$$

$$M[\psi_1(k)]_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 + \theta_{r,n_1}C_X^2\left(1-k_{YX}\right) - \theta_{n_1,N}k_{YZ}^2C_Z^2\right\}$$

$$M[\psi_2(k)]_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 + \theta_{r,n_1}C_X^2\left(1-k_{YX}\right) - \theta_{r,n_1}\left(k_{YZ} - k_{XZ}\right)^2C_Z^2\right\}$$

$$M[\psi_3(k)]_{min} = \overline{Y}^2\left\{\theta_{r,N}C_Y^2 + \theta_{r,n_1}C_X^2\left(1-k_{YX}\right) - \left(\theta_{n_1,N}k_{YZ} - \theta_{r,n_1}k_{XZ}\right)^2\left(\theta_{r,N}\right)^{-1}C_Z^2\right\}$$

Where $\qquad \phi_1 = \dfrac{fB}{A+fB+C} \qquad \phi_2 = \dfrac{C}{A+fB+C} \qquad \phi = \phi_1 - \phi_2 = \dfrac{fB-C}{A+fB+C}$

From the optimum value of ϕ one can find the optimum value or values of k. Using this value of k, the optimum values of A, B, C and ϕ_2 can be calculated.

6 Theoretical comparison of the proposed point estimator with existing estimators

In this section we derive the conditions under which the suggested estimator is superior to the existing estimators. To derive the conditions, consider the following notation.

$A_X = \overline{Y}C_X \qquad A_Y = \overline{Y}C_Y \qquad A_{YX} = \overline{Y}\rho_{YX}C_X \qquad A_{YZ} = \overline{Y}\rho_{YZ}C_Z$

6.1 $Comparison\ with\ Mean\ Method\ of\ Imputation$

$$V(\overline{y}_m) - M(\overline{y}_{GS}^{dc})_{min} = \left(\theta_{r,n_1}A_{YX}^2 + \theta_{n_1,N}A_{YZ}^2\right) > 0$$

Thus the proposed point estimator is more efficient than $\overline{y}_m$.

6.2 $Comparison\ with\ Ratio\ Method\ of\ Imputation$

$$M(\overline{y}_R) - M(\overline{y}_{GS}^{dc})_{min} = \left\{\theta_{r,n}\left(A_X - A_{YX}\right)^2 + \theta_{r,n_1}A_{YX}^2 + \theta_{n_1,N}A_{YZ}^2\right\} > 0$$

Thus the proposed point estimator is more efficient than $\overline{y}_R$.

6.3 $Comparison\ with\ Compromised\ Method\ of\ Imputation$

$$M(\overline{y}_{Comp})_{min} - M(\overline{y}_{GS}^{dc})_{min} = \left\{\theta_{r,n_1}A_{YX}^2 + \theta_{n_1,N}A_{YZ}^2\right\} > 0$$

Thus the proposed point estimator is more efficient than $\overline{y}_{Comp}$.

6.4 $Comparison\ with\ Ratio\ cum\ Product\ compromised\ Method\ of\ Imputation$

$$M(\overline{y}_{IRP1}^d)_{min} - M(\overline{y}_{GS}^{dc})_{min} = \left\{\theta_{r,n}A_{YX}^2 + \theta_{n_1,N}A_{YZ}^2\right\} > 0$$
$$M(\overline{y}_{IRP2}^d)_{min} - M(\overline{y}_{GS}^{dc})_{min} = \left\{\theta_{r,n_1}A_{YX}^2 + \theta_{n_1,N}A_{YZ}^2\right\} > 0$$
$$M(\overline{y}_{IRP3}^d)_{min} - M(\overline{y}_{GS}^{dc})_{min} = \theta_{n_1,N}A_{YZ}^2 > 0$$

Thus the proposed point estimator is more efficient than $\overline{y}_{IRP1}^d$, $\overline{y}_{IRP2}^d$ and $\overline{y}_{IRP3}^d$.

6.5 $Comparison\ with\ Exponential\ chain\ ratio\ Method\ of\ Imputation$

$$M(\overline{y}_{IRe}^{dc})_{min} - M(\overline{y}_{GS}^{dc})_{min} = Q^* \left(A_{YX}C_Z - A_{YZ}C_X\right)^2 > 0$$

Where $Q^* = \frac{\theta_{n_1,N}\theta_{r.n_1}}{\theta_{r,n_1}C_X^2 + \theta_{n_1,N}C_Z^2} > 0$

Thus the proposed point estimator is more efficient than $\overline{y}_{IRe}^{dc}$.

6.6 $Comparison\ with\ factor\ type\ chain\ Method\ of\ Imputation$

$$M[\psi_1(k)]_{min} - M(\overline{y}_{GS}^{dc})_{min} = A_X^2\theta_{r,n_1}\left(1 - k_{YX}\right)^2 > 0$$

Thus the proposed point estimator is more efficient than $\psi_1(k)$.

$$M[\psi_2(k)]_{min} - M(\overline{y}_{GS}^{dc})_{min} = A_X^2\theta_{r,n_1}\left(1 - k_{YX}\right)^2 + A_{YZ}^2\theta_{n_1,N} - A_Z^2\theta_{r,n_1}\left(k_{YZ} - k_{XZ}\right)^2$$

Thus the proposed point estimator is more efficient than $\psi_2(k)$ if

$$A_X^2\theta_{r,n_1}\left(1 - k_{YX}\right)^2 + A_{YZ}^2\theta_{n_1,N} > A_Z^2\theta_{r,n_1}\left(k_{YZ} - k_{XZ}\right)^2 \implies F_1 > F_2$$

$$M[\psi_3(k)]_{min} - M(\overline{y}_{GS}^{dc})_{min} = A_X^2\theta_{r,n_1}\left(1 - k_{YX}\right)^2 + A_{YZ}^2\theta_{n_1,N} - A_Z^2\left(\theta_{r,N}k_{YZ} - \theta_{r,n_1}k_{XZ}\right)^2\left(\theta_{r,N}\right)^{-1}$$

Thus the proposed point estimator is more efficient than $\psi_3(k)$ if

$$A_X^2\theta_{r,n_1}\left(1 - k_{YX}\right)^2 + A_{YZ}^2\theta_{n_1,N} > A_Z^2\left(\theta_{r,N}k_{YZ} - \theta_{r,n_1}k_{XZ}\right)^2\left(\theta_{r,N}\right)^{-1} \implies F_1 > F_3$$

7 Practicability and limitations of the proposed Imputation Methods

The main difficulty in using the proposed point estimator $\overline{y}_{GS}^{dc}$ is the availability of α and β as the optimum values of α and β . Thus, if the values of ρ_{YX} , ρ_{YZ}, C_Y, $C_X$and C_Z are known from the past survey, then the proposed estimator may be used for estimation of finite population mean on current occasion. Sometimes the values of the above mentioned parameters may not be known and even enough founds may not available for conducting a pilot survey. Under such circumstance it is always preferable to replace the optimum values of α and β with their respective sample estimates. Thus, we have

$$\hat{\alpha}_{opt} = \frac{2}{g}r_{yx}\frac{c_y}{c_x} \qquad \text{and} \qquad \hat{\phi}_{opt} = \frac{r_{yz}c_x}{r_{yx}c_z}$$

Where r_{yx}, r_{yz} and r_{xz} are the sample correlation coefficient between Y and X, Y and Z, and X and Z respectively. c_y, c_x and c_z are the sample coefficients of standard deviation of the variable Y, X and Z respectively. All the above statistics are calculated from the responding units of the sample S_1 or S.

The main limitation of the proposed point estimator is that, if the necessary parameters are not known before the survey, then it is preferable to use corresponding sample estimates, which may decrease the accuracy of the result. Another limitation is that if the population is not homogeneous then also the proposed imputation method may fail to give efficient estimate of the missing values in the sample.

7.1 Practical Illustration

In this section, we have illustrated a practical situation where an incomplete data set is completed with the help of imputation techniques. Consider a population of size 33 [Cochran, W (1977) Page No. 34]. Here Y is the food cost of the family, X is the income of the family, and Z is the size of the family. A preliminary sample S_1 of size 16 selected randomly from the population and a sample S of size 10 selected randomly from the sample . The data obtained given in the form of three tuples is (14.3, 62, 2), (63.0, 95, 6), (– , 77,3), (20.8, 62, 3), (–, 87, 3), (–, 65, 5), (37.8, 65, 7), (–, 63, 2), (41.2, 58, 4), (39.8, 77,3), (–, 88, 2), (–, 79, 4), (44.4, 62, 5), (29.4, 60, 4),(–, 73, 4) (36.0, 83, 4). The sample of size 10 selected randomly was found to have 4 missing values corresponding to the study variable Y. The data obtained given in the form of three tuples is (14.3, 62, 2), (20.8, 62, 3), (–, 87, 3), (–, 65, 5), (41.2, 58, 4), (–, 88, 2), (–, 79, 4), (44.4, 62, 5), (29.4, 60, 4), (36.0, 83, 4). It is assumed that corresponding to the given data set, the values of the parameters ρ_{YX}, ρ_{YZ}, ρ_{XZ},C_Y, C_X and C_Z are known. If these values are not known in advance, then the values of ρ_{YX}, ρ_{YZ}, ρ_{XZ},C_Y, C_X and C_Z are estimated from the responding group of the sample S_1 . Here, $\overline{Z} = 3.730$, $\rho_{YX} = 0.2525$, $\rho_{YZ} = 0.433$, $C_Y = 0.368$, $C_X = 0.146$, $C_Z = 0.409$ From the sample S , we have $\overline{y}_r = 31.02$, $\overline{x}_r = 64.5$, $\overline{x}_n = 70.6$ and $\overline{z}_n = 3.6$. And from the sample S_1, we have $\overline{x}_1 = 72.25$ and $\overline{z}_1 = 3.81$ As per the Mean method of imputation, missing values y_3, y_4, y_6 and y_7 are replaced by $\overline{y}_r = 31.02$. As per Ratio method of imputation, missing values are replaced by $\hat{y}_3 = \frac{\overline{y}_r}{\overline{x}_r} \times x_3 = \frac{\overline{y}_r}{\overline{x}_r} \times 87 = 41.84$, $\hat{y}_4 = \frac{\overline{y}_r}{\overline{x}_r} \times x_4 = \frac{\overline{y}_r}{\overline{x}_r} \times 65 = 31.26$, $\hat{y}_6 = \frac{\overline{y}_r}{\overline{x}_r} \times x_6 = \frac{\overline{y}_r}{\overline{x}_r} \times 79 = 37.99$, and $\hat{y}_7 = \frac{\overline{y}_r}{\overline{x}_r} \times x_7 = \frac{\overline{y}_r}{\overline{x}_r} \times 88 = 42.32$. Now, the optimum values of the constraints used in the proposed imputation method are

$\alpha_{opt} = \frac{2}{g}\rho_{YX}\frac{C_Y}{C_X} = 0.76206$, $\phi_{opt} = \frac{\rho_{YZ}C_X}{\rho_{YX}C_Z} = 0.61336$ and $\frac{b}{a} = \left(\frac{1}{\phi_{opt}-1}\right)\overline{Z} = 2.35126$
Thus,

$$\zeta = \left\{ \frac{\frac{n_1\overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)}{\frac{n_1\overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\left(\frac{a\overline{Z}+b}{a\overline{z}_1+b}\right)} \right\} = \left\{ \frac{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)} - \overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}}{\frac{n_1\overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}-n\overline{x}_r}{(n_1-n)} + \overline{x}_1\frac{\overline{Z}^*}{\overline{z}_1^*}} \right\} = \frac{11.329556}{153.922304} = 0.073605681$$

and $\exp(\alpha_{opt}\zeta) = 1.05769493$
Hence, as per the imputation method proposed, the missing values are replaced by the following value given by

$$\frac{\overline{y}_r}{n-r}\left\{n\exp\left(n\alpha_{opt}\zeta\right) - r\right\} = 34.0028278 \simeq 34.003$$

8 Empirical Study

To illustrate the performance of the proposed point estimators of the finite population mean, we have considered two population data set from the literature. The percentage relative efficiencies of the proposed point estimator with respect to the Mean, Ratio and Compromised methods of imputation are given as

$$PRE = \frac{M(\overline{y}_m)}{M(\cdot)} \times 100, \quad PRE_I = \frac{M(\overline{y}_R)}{M(\cdot)} \times 100 \quad \text{and} \quad PRE_{II} = \frac{M(\overline{y}_{Comp})}{M(\cdot)} \times 100$$

The conditions for supremacy of the proposed point estimator over the existing factor type chain ratio estimators $\psi_2(k)$ and $\psi_3(k)$for the two population I and II have also been verified numerically.

Population I: The data has been taken from Sukhatme and Chand (1977) which was reproduced in Singh et al. (1994). The description of the population as follows; Y,X and Z are the apple trees of bearing age in 1964, Bushels of apples harvested in 1964 and Bushels of apples harvested in 1959, respectively.
The combination of N, n_1, n and r are 200, 70, 30 and 22.

Population II: This population was artificially generated for three variables Y, X and Z by Shukla and Thakur (2008). Considering Y as study variable and X and Z as the main and additional auxiliary variables, respectively.
The combination of N, n_1, n and r are 200, 70, 30 and 22.

$Parameters$	$PopulationI$	$PopulationII$
$\overline{Y}$	1031.82	42.485
$\overline{X}$	2934.58	18.515
$\overline{Z}$	3651.49	20.52
ρ_{YX}	0.9300	0.8734
ρ_{YZ}	0.7700	0.8667
ρ_{XZ}	0.8400	0.9943
C_Y^2	2.55280	0.10804
C_X^2	4.025045	0.1410
C_Z^2	2.09379	0.10864

Table 1: Parameters of the Population I and II

From the subsection **6.6**, we get numerical values of the conditions for the Population I are

$$F_1 = A_X^2 \theta_{r,n_1} \left(1 - k_{YX}\right)^2 + A_{YZ}^2 \theta_{n_1,N} = 146130.9766 \qquad F_2 = A_Z^2 \theta_{r,n_1} \left(k_{YZ} - k_{XZ}\right)^2 = 7268.0190$$
$$F_3 = A_Z^2 \left(\theta_{r,N} k_{YZ} - \theta_{r,n_1} k_{XZ}\right)^2 \left(\theta_{r,N}\right)^{-1} = 5431.9938$$

And for the Population II are

$$F_1 = A_X^2 \theta_{r,n_1} \left(1 - k_{YX}\right)^2 + A_{YZ}^2 \theta_{n_1,N} = 6.363024 \qquad F_2 = A_Z^2 \theta_{r,n_1} \left(k_{YZ} - k_{XZ}\right)^2 = 4.053797$$
$$F_3 = A_Z^2 \left(\theta_{r,N} k_{YZ} - \theta_{r,n_1} k_{XZ}\right)^2 \left(\theta_{r,N}\right)^{-1} = 0.000562$$

It is observed that $F_1 > F_2$ and $F_1 > F_3$. Thus, the proposed point estimator is more efficient than the point estimators $\psi_2(k)$ and $\psi_3(k)$ for both population I and II.

$Point\ \ Estimator$	$PopulationI$	$PopulationII$
$\overline{y}_m$	109949.12418	7.889021625
$\overline{y}_R$	84950.34351	6.2565842223
$\overline{y}_{Comp}$	81456.24103	6.085886608

Table 2: MSE of the point estimators $\overline{y}_m$, $\overline{y}_R$ and $\overline{y}_{Comp}$

$Point\ \ Estimator$	$PopulationI$	$PopulationII$
t_1	39575.35244	3.095732593
t_2	21933.84130	1.906742346
t_3	44250.01204	3.531679331
t_4	42231.33809	3.407014655
t_5	48993.37058	3.424484781
t_6	45916.64953	3.108978931
t_7	47436.69184	3.931661228
t_8	55518.85851	3.954054268
t_9	51846.24046	3.548742805
t_{10}	22168.75005	1.944613618
$\overline{y}_{GS}^{dc}$	21718.62288	1.892170059

Table 3: MSE of the proposed point estimators and its particular cases

Point Estimator	Population I		Population II	
	PRE_I	PRE_{II}	PRE_I	PRE_{II}
t_1	214.654673	205.825687	202.1035097	196.589545
t_2	387.302627	371.372437	328.129505	319.177188
t_3	191.978125	184.081851	177.156068	172.322740
t_4	201.154752	371.372437	328.129505	319.177188
t_5	173.391507	166.259721	182.701475	177.716854
t_6	185.009892	177.400228	201.242413	195.751941
t_7	179.081509	171.715686	159.133350	154.791734
t_8	153.011690	146.718148	158.232128	153.915101
t_9	163.850537	157.111182	176.304245	171.494158
t_{10}	383.198617	367.437230	321.739196	312.961225
$\overline{y}_{GS}^{dc}$	391.140562	375.05251	330.656549	321.635287

Table 4: PRE of the proposed point estimators and its particular cases

Increasing or decreasing the percentage of non-responses in the sample affect the MSE and PRE of a point estimator. Thus we calculate MSE and PRE of the different estimators for population I by considering the non-responses $(r^* = n - r)$ between 25% to 60% in the sample S

Point Estimator	r=20 and r$^* = 10$	r=17 and r$^* = 13$	r=15 and r$^* = 15$	r=12 and r$^* = 18$
$\overline{y}_m$	122302.95836	146283.930596	167600.350355	212897.742343
$\overline{y}_R$	87929.63464	93712.965357	98853.703504	109777.772067
$\overline{y}_{Comp}$	83125.24404	86365.073384	89244.921694	95364.599352
t_1	43848.461838	52143.321246	59516.529609	75184.597382
t_2	23606.583526	26851.705749	29734.888251	35859.300491
t_3	48523.121435	56817.980843	64191.189207	79859.256979
t_4	46891.778722	55938.516420	63980.061041	81068.343358
t_5	54639.940945	65600.930487	75344.032301	96048.123657
t_6	51114.531417	61204.537427	70173.431659	89232.331901
t_7	52097.132474	61143.870172	69185.414793	86273.697111
t_8	61165.428885	72126.418426	81869.520241	102573611596
t_9	57044.122346	67134.128357	76103.022588	95161.922830
t_{10}	23841.388955	27086.361955	29969.448665	36093.722504
$\overline{y}_{GS}^{dc}$	23387.625878	26627.455226	29507.303536	35626.981193

Table 5: MSE of the proposed and existing point estimators for Population I

Point Estimator	r* = 8	r* = 10	r* = 13	r* = 15	r* = 18
$\overline{y}_m$	100.000	100.000	100.000	100.000	100.000
$\overline{y}_R$	129.427	139.092	156.098	169.544	193.935
$\overline{y}_{Comp}$	134.979	147.131	169.378	187.798	223.246
t_1	277.822	278.922	280.542	169.542	283.167
t_2	501.276	518.088	544.784	563.649	593.703
t_3	248.473	252.051	257.461	261.096	266.591
t_4	260.349	260.8196	261.508	261.957	262.615
t_5	224.416	223.834	222.990	222.446	221.657
t_6	239.454	239.272	239.008	238.837	238.588
t_7	231.781	234.759	239.245	242.248	246.770
t_8	198.039	199.954	202.816	204.716	207.556
t_9	212.067	214.401	217.898	220.228	223.722
t_{10}	495.964	512.986	540.065	559.237	589.847
$\overline{y}_{GS}^{dc}$	506.243	522.938	549.372	567.996	597.574

Table 6: PRE of the proposed and existing point estimators for Population I

Variation of the correlation coefficient or coefficients will affect the MSE and PRE of a ratio type estimators. Thus we calculate MSE and PRE of the point estimators for different values of the correlation coefficients ρ_{YX} and ρ_{YZ}.

Point Estimator	$\rho_{YX} = 0.94$ $\rho_{YZ} = 0.80$	$\rho_{YX} = 0.95$ $\rho_{YZ} = 0.83$	$\rho_{YX} = 0.96$ $\rho_{YZ} = 0.86$
$\overline{y}_m$	109949.124188	109949.124188	109949.124188
$\overline{y}_R$	84123.016391	83295.689268	82468.362145
$\overline{y}_{Comp}$	80840.196508	80217.563265	79588.341312
t_1	37588.904442	35537.029627	33479.727998
t_2	19271.238008	16568.958596	13827.003074
t_3	42937.975057	41625.938072	40303.901086
t_4	39902.060652	37515.157373	35070.628252
t_5	45825.344389	42597.833701	39310.838509
t_6	44057.846286	42153.313227	40203.959354
t_7	45735.120901	44011.647955	42266.273004
t_8	52869.217354	50179.902585	47450.914205
t_9	50744.129394	49642.018326	48539.907259
t_{10}	19523.523941	16839.030869	14115.270838
$\overline{y}_{GS}^{dc}$	18945.840096	16110.688097	13213.166884

Table 7: MSE of the proposed and existing point estimators for Population I

Point Estimator	$\rho_{YX}=0.93$ $\rho_{YZ}=0.77$	$\rho_{YX}=0.94$ $\rho_{YZ}=0.80$	$\rho_{YX}=0.95$ $\rho_{YZ}=0.83$	$\rho_{YX}=0.96$ $\rho_{YZ}=0.86$
$\overline{y}_R$	100.000	100.000	100.000	100.000
$\overline{y}_{Comp}$	104.289	104.061	103.837	103.618
t_1	214.654	233.797	234.391	246.323
t_2	287.387	436.521	502.721	596.429
t_3	191.978	195.917	200.105	204.616
t_4	201.154	210.824	222.032	235.149
t_5	173.391	183.573	195.539	209.785
t_6	185.009	190.938	197.602	205.124
t_7	179.081	183.935	189.258	195.116
t_8	153.011	151.115	165.994	173.797
t_9	163.850	165.779	167.793	169.898
t_{10}	383.198	430.880	494.658	584.249
$\overline{y}_{GS}^{dc}$	391.140	444.018	517.021	624.138

Table 8: PRE_I of the proposed and existing point estimators for Population I

9 Simulation Study

For simulation study, we use the data set of the population II of size $N = 200$ as attached in Appendix A. The steps followed for the simulation procedure are:
Step 1: A random sample S_1 of size $n_1 = 80$ is selected from the population

Step 2: A sub-sample $S$of size n is selected from S_1 selected in step 1.

Step 3: From each sample S selected in step 2, $(n - r)$ units are drop away randomly corresponding to the study variable Y only.

Step 4: Compute and impute the dropped units of Y with the help of existing and proposed imputation methods.

Step 5: Calculate the estimates of the population mean for existing and proposed imputation methods.

Step 6: Repeat Step (1 to 5) $k = 500,000$ times, which provides k times sample-based estimates $\hat{T}_1$,$\hat{T}_1$,$\hat{T}_2$,$\hat{T}_3 \cdots \hat{T}_{500000}$.
Step 7: The $Bias$ and MSE of T are obtained by

$$B(\hat{T}) = \frac{1}{k}\sum_{i=1}^{N}\left(\hat{T}-\overline{Y}\right) \qquad\qquad M(\hat{T}) = \frac{1}{k}\sum_{i=1}^{N}\left(\hat{T}-\overline{Y}\right)^2$$

Step 8: The PRE of with respect to Mean, Ratio and Compromised imputation method are

$$PRE = \frac{M\left(\overline{y}_m\right)}{M\left(T\right)}\times 100 \qquad PRE_I = \frac{M\left(\overline{y}_R\right)}{M\left(T\right)}\times 100 \qquad PRE_{II} = \frac{M\left(\overline{y}_{Comp}\right)}{M\left(T\right)}\times 100$$

The well-known software Wolfram Mathematica is used to run the above simulation programe numerical values of Bias, MSE and PRE of the different point estimators are as follows:

Point Estimator	$Bias$	MSE
$\overline{y}_m$	0.001375	7.89195
$\overline{y}_R$	0.015983	6.27867
$\overline{y}_{Comp}$	0.011895	6.10253

Table 9: $Bias$ and MSE of the point estimators $\overline{y}_m$, $\overline{y}_R$ and $\overline{y}_{Comp}$

Point Estimator	$Bias$	MSE
t_1	-0.117659	3.11660
t_2	-0.141049	1.96844
t_3	-0.086101	3.52122
t_4	-0.115447	3.42049
t_5	-0.115358	3.43756
t_6	-0.117544	3.012953
t_7	-0.077201	3.91559
t_8	-0.076715	3.93772
t_9	-0.085712	3.53800
t_{10}	-0.1404493	1.97206
$\overline{y}_{GS}^{dc}$	0.014290	1.95984

Table 10: $Bias$ and MSE of the proposed point estimators and its particular cases

Point Estimator	PRE_I	PRE_{II}
t_1	201.459	195.807
t_2	318.968	310.019
t_3	178.310	173.307
t_4	183.561	178.411
t_5	182.649	177.525
t_6	200.627	194.998
t_7	160.351	155.852
t_8	159.450	154.978
t_9	177.464	172.485
t_{10}	318.383	309.449
$\overline{y}_{GS}^{dc}$	320.368	311.379

Table 11: PRE of the proposed point estimators and its particular cases

10 Conclusion

The mean square error of the proposed estimators are derived in terms of population parameters and compared the MSE of the proposed point estimator $\overline{y}_{GS}^{dc}$ with the MSE of existing point estimators $\overline{y}_m$, $\overline{y}_R$, $\overline{y}_{Comp}$, $\overline{y}_{IRP1}^{d}$, $\overline{y}_{IRP2}^{d}$, $\overline{y}_{IRP3}^{d}$, $\overline{y}_{IRe}^{dc}$, $\psi_1(k)$, $\psi_2(k)$ and $\psi_3(k)$. It is found theoretically that the proposed estimator has lesser MSE than the existing point estimators taken into comparisons except two conditions $F_1 > F_2$ and $F_1 > F_3$. This two conditions are verified numerically and are satisfied for both Population I and II. We have also compared empirically the MSE and PRE of the proposed estimators by using two population data sets and found that the proposed point estimator $\overline{y}_{GS}^{dc}$ has better PRE in case of both the Populations I and II. Simulation studies have been carried out to verify the validity of the theoretical findings of the proposed and existing estimators. Both the numerical values of theMSE and PRE of the proposed point estimator and the point estimators of the particular cases found from the theoretical results and the simulation results are similar. The numerical values of MSE and PRE of the point estimators are also calculated for the different values of non-responses r^* between 25% and 60% and different values of the correlation coefficients for population I are given in Table 5 to Table 8. It is observed that PRE of the proposed point estimator increases as the number of non-responses increases. It is also observed that PRE of the proposed point estimator increases as the correlation coefficients increases. Thus, proposed estimator is preferable in use over other estimators.

11 Scope of further Investigation

The proposed imputation method can be described in stratified random sampling for improve estimation of the finite population mean of the study variable, if the population is not homogeneous with respect to the study variable.

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13 Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this research paper.

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Appendix A

Y	45	50	39	60	42	38	28	42	38	35	40	55	45	36	40	58	56	62	58	46
X	15	20	23	35	18	12	08	15	17	13	29	35	20	14	18	25	28	21	19	18
Z	16	22	26	37	19	14	11	17	18	15	30	37	23	15	19	27	30	22	21	21
Y	36	43	68	70	50	56	45	32	30	38	35	41	45	65	30	28	32	38	61	58
X	15	20	38	42	23	25	18	11	09	17	13	15	18	25	09	08	11	13	23	21
Z	18	22	39	44	25	26	19	13	12	20	16	17	19	27	12	10	13	14	24	23
Y	65	62	68	85	40	32	60	57	47	55	67	70	60	40	35	30	25	38	23	55
X	27	25	30	45	15	12	22	19	17	21	25	30	27	21	15	17	09	15	11	21
Z	28	26	33	46	17	15	23	20	19	23	26	32	30	23	17	18	12	18	14	24
Y	50	69	53	55	71	74	55	39	43	45	61	72	65	39	43	57	37	71	71	70
X	15	23	29	30	33	31	17	14	17	19	25	31	30	19	21	23	15	30	32	29
Z	17	24	30	33	35	32	19	16	19	21	27	33	32	21	23	25	17	32	33	32
Y	73	63	67	47	53	51	54	57	59	39	23	25	35	30	38	60	60	40	47	30
X	28	23	23	17	19	17	18	21	23	20	07	09	15	11	13	25	27	15	17	11
Z	30	25	24	20	22	20	21	23	26	22	10	11	18	14	14	26	29	18	20	14
Y	57	54	60	51	26	32	30	45	55	54	33	33	20	25	28	40	33	38	41	33
X	31	23	25	17	09	11	13	19	25	27	13	11	07	09	13	15	13	17	15	13
Z	32	25	27	19	12	13	14	20	27	28	16	14	09	10	14	17	14	20	17	15
Y	30	35	20	18	20	27	23	42	37	45	37	37	37	34	41	35	39	45	24	27
X	11	15	08	07	09	13	12	25	21	22	15	16	17	13	20	15	21	25	11	13
Z	13	18	11	08	12	16	14	26	24	23	16	18	19	16	22	18	23	26	14	14
Y	23	20	26	26	40	56	41	47	43	33	37	27	21	23	24	21	39	33	25	35
X	09	08	11	12	15	25	15	25	21	15	17	13	11	11	09	08	15	17	11	19
Z	11	10	14	15	17	26	17	27	22	17	19	16	13	12	12	11	17	20	13	20
Y	45	40	31	20	40	50	45	35	30	35	32	27	30	33	31	47	43	35	30	40
X	21	23	15	11	20	25	23	17	16	18	15	13	14	17	15	25	23	17	16	19
Z	22	25	18	13	21	27	26	19	17	19	17	16	16	14	17	28	25	18	18	22
Y	35	35	46	39	35	30	31	53	63	41	52	43	39	37	20	23	35	39	45	37
X	19	19	23	15	17	13	19	25	35	21	25	19	18	17	11	09	15	17	19	19
Z	22	21	24	17	20	15	22	26	36	23	26	20	20	19	13	12	17	18	21	22