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Bayesian Estimation of a Rare Sensitive Characteristic in a Stratified Sample using a Non-Gamma Prior

Samridhi Mehta[1], **Priyanka Aggarwal**[2]
[1]Department of Mathematics, Hindu College, University of Delhi, Delhi – 110007, India
[2]Department of Statistics, Hindu College, University of Delhi, Delhi – 110007, India
Corresponding author: Priyanka Aggarwal, *E-mail: priyankaaggarwal@hinducollege.ac.in*

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Abstract

This study deals with the extension of the Binary Optional Unrelated Question Model of Sihm et al. (2016) into a stratified set up wherein the prevalence of sensitive characteristic is rare. The Bayes estimator of the sensitivity level is obtained using Beta prior and Squared Error Loss Function. Also, the Bayes estimator of the prevalence of rare sensitive attribute is obtained using a class of non-gamma priors and Linex Loss Function. The corresponding minimal values of Bayes posterior expected loss (BPEL) are also derived. Relative losses are also examined to compare the Bayes estimates with those of the classical estimates. The robustness of the Bayes estimator of the prevalence of rare sensitive attribute is studied with respect to a moderately non-gamma prior distribution and for a class of Linex Loss Function via a numerical illustration.

Keywords: Binary unrelated question randomized response model, Stratified survey sampling, Bayesian estimation, Linex loss function, Relative losses.

1 Introduction

The randomized response technique (RRT), a pioneering indirect questioning method, has been widely employed in survey research for over six decades to collect sensitive data across various disciplines, including behavioral science, social, economic, epidemiological, psychological, biomedical among others. Recently, Mehta et al. (2024) used a binary RRT model to estimate the prevalence of 'having a troubled childhood' among college students. Despite its longevity, the technique has undergone significant improvements, with numerous variations and enhancements emerging in the literature since its introduction by Stanley Warner in 1965. Initially designed to increase response rates and eliminate dishonest responses, RRT aims to estimate the proportion of individuals with sensitive attributes while protecting respondent privacy. The technique adds random noise to respondents' answers, ensuring confidentiality. Le et al. (2023) provides a comprehensive systematic review of randomized response techniques, tracing their evolution from Warner's (1965) seminal work to the current state of the field.
A binary optional unrelated model was discussed in Sihm et al. (2016). This model also uses the unrelated question model that of Greenberg et al. (1969) and binary optional model of Gupta et al. (2013). In this paper, binary model of Sihm et al. (2016) is extended to stratified sampling of a rare sensitive attribute under the assumption of known strata sizes. Earlier, Kim & Warde (2004) introduced a stratified version of Warner's (1965) randomized response model. Lee et al. (2013) explored stratified sampling of a rare sensitive attribute as an extension of Land et al. (2011). Implementing a stratified sampling scheme allows researchers to enhance the sensitivity and relevance of the information collected, resulting in a deeper understanding of complex issues. This approach not only boosts the efficiency of data collection but also ensures that diverse groups are represented, contributing to more informed decision-making and

effective solutions. The extension of Sihm et al. (2016) model to a three-stage model can be seen in Chhabra et al. (2016).

The well-known approach of Bayesian Estimation involves using the available prior information about the unknown parameter along with the sample information for its estimation. Literature suggests that researchers have been exploring the use of Bayesian analysis in estimating the prevalence of a sensitive characteristic through RRT models. Hussain & Shabbir (2012) implemented a stratified random sampling approach combined with the Bayesian method to estimate the population proportion of a sensitive characteristic. Adeptun & Adewara (2014) proposed Bayesian estimators for the population proportion of a stigmatized feature, utilizing both Kumaraswamy and generalized beta prior distributions with data collected through the randomized response technique (RRT) of Kim & Warde (2004). Song & Kim (2017) also applied the RRT to develop Bayesian estimates for the prevalence of a rare sensitive trait. Additionally, Mehta & Aggarwal (2018) and Narjis & Shabbir (2021) provided Bayesian estimates for the sensitivity level and population proportion of sensitive attributes using optional unrelated question random response models.

In this paper, squared error loss function (SELF) and natural conjugate Beta prior distribution are used to obtain Bayes estimator of the sensitivity level of the stratified version of binary Sihm et al. (2016) randomized response model. Also, Khamis' class (K-class) of moderately non-gamma prior distributions for the unknown prevalence of rare sensitive attribute is used in this paper and its Bayes estimator is derived under Varian's (1975) asymmetric Linex loss function.

Berger (1984) observed that subjectivity involved in selecting a single prior distribution and a loss function had drawn severe criticism to Bayesian methodology. Therefore, sensitivity of Bayes predictors to a moderate violation of assumptions concerning prior distribution and loss function must be examined. In this paper, we follow Bansal & Singh (1999) and Aggarwal & Bansal (2017) to use Khamis' class (K-class) of moderately non-gamma prior distributions and study the robustness of the Bayes estimate with respect to the prior. Furthermore, the robustness of the Bayes estimate is examined with respect to a moderate change in the shape parameter of Linex loss function.

The stratified version of Sihm et al. (2016) randomized response model is described in Section 2 which is followed by the classical estimation of the prevalence of sensitive attribute and the sensitivity level in Section 3. In Section 4, the corresponding Bayesian estimators are derived. The comparative study of the Bayesian estimators and the classical estimators is presented in Section 5 using relative losses. Section 6 examines the sensitivity of Bayes predictor of the prevalence of sensitive attribute to a possible misspecification of the loss function and with regard to a moderately non-gamma prior distribution. The conclusion is given in Section 7 followed by references and appendices.

2 Stratified Sihm et al. (2016) model

Sihm et al. (2016) introduced a modification to the optional unrelated question models developed by Gupta et al. (2013) for binary responses. Their approach eliminates the need for a split sample, which typically demands a larger overall sample size. Instead of attempting to estimate prevalence and sensitivity at the same time, they estimated these metrics independently by asking two distinct questions. First, respondents answer an auxiliary question to determine whether they consider the underlying research question sensitive enough to warrant a scrambled response. This auxiliary question is based on the original model proposed by Greenberg et al. (1969), allowing for the estimation of the proportion of respondents who view the main research question as sensitive (ω). The second question addresses the main research inquiry, with respondents using the optional model from Gupta et al. (2013) to estimate the proportion of the population that possesses the sensitive characteristic (π).

In this paper, the above model is extended to stratified survey sampling of a rare sensitive attribute (π) under the assumption that the sizes of each strata is known. Let a population of size N be divided into disjoint L strata of size $N_h (h = 1, 2, \ldots, L)$. In stratum h, let

n_h be the sample size,

ω_h be the sensitivity level of the main research question,
π_h be the unknown proportion of population that possesses the rare sensitive attribute,
π_{bh} be the known probability of an unrelated question in randomization device 1,
p_{bh} be the known probability of the respondent being asked if the main research question is sensitive for him/her in randomization device 1,
π_{ah} is the known probability of another unrelated question describing a rare attribute in randomization device 2,
p_{ah} is the known probability of the respondent being asked the main research question as the sensitive question in randomization device 2.
The probability θ_h of a 'Yes' response from a respondent in randomization device 1 in h^{th} stratum is

$$\theta_h = p_{bh}\omega_h + (1 - p_{bh})\pi_{bh} \tag{1}$$

Let

$$y_{hi} = \begin{cases} 1, & with\ probability\ \theta_h \\ 0, & with\ probability\ 1 - \theta_h \end{cases}$$

where θ_h is defined in (1).
Let $Y_h = \sum_{i=1}^{n_h} y_{hi}$ represent the total number of yes responses in randomization device 1 in a sample of size n_h drawn from the population in the h^{th} stratum using simple random sampling with replacement. The sampling process is Bernoulli in θ_h and hence, the likelihood function is

$$\ell\left(\theta_h|y_h\right) = \binom{n_h}{y_h} \theta_h^{y_h} \left(1 - \theta_h\right)^{n_h - y_h},\ 0 < \theta_h < 1$$

In terms of ω_h

$$\begin{aligned}
\ell\left(\omega_h|y_h\right) &= \binom{n_h}{y_h} \left(p_{bh}\omega_h + (1 - p_{bh})\,\pi_b\right)^{y_h} \left(1 - p_{bh}\omega_h - (1 - p_{bh})\,\pi_b\right)^{n_h - y_h} \\
&= \binom{n_h}{y_h} p_{bh}^{y_h + n_h - y_h} \left(\omega_h + \frac{(1 - p_{bh})\,\pi_b}{p_{bh}}\right)^{y_h} \left(1 - \omega_h + \frac{(1 - p_{bh})\,(1 - \pi_b)}{p_{bh}}\right)^{n_h - y_h} \\
&= \binom{n_h}{y_h} p_{bh}^{n_h} \left(\omega_h + f_h\right)^{y_h} \left(1 - \omega_h + k_h\right)^{n_h - y_h}
\end{aligned}$$

where $f_h = \dfrac{(1 - p_{bh})\pi_b}{p_{bh}}$ and $k_h = \dfrac{(1 - p_{bh})(1 - \pi_b)}{p_{bh}}$.
Using binomial expansions of $(\omega_h + f_h)^{y_h}$ and $(1 - \omega_h + k_h)^{n_h - y_h}$, we get

$$\ell\left(\omega_h|y_h\right) = \binom{n_h}{y_h} p_{bh}^{n_h} \left(\sum_{i=0}^{y_h} \binom{y_h}{i} \omega_h^i f_h^{y_h - i}\right) \left(\sum_{j=0}^{n_h - y_h} \binom{n_h - y_h}{j} \left(1 - \omega_h\right)^j k_h^{n_h - y_h - j}\right) \tag{2}$$

The probability ϕ_h of a 'Yes' response from a respondent in randomization device 2 in h^{th} stratum is

$$\phi_h = \left(1 - \omega_h\right)\pi_h \ +\ \omega_h\left(p_{ah}\pi_h + (1 - p_{ah})\pi_{ah}\right) \tag{3}$$

Since both sensitive attribute and the unrelated question of randomization device 2 are rare, $n_h\phi_h = \lambda_h$ (finite), assuming $n_h \to \infty$ and $\phi_h \to 0$. Let $y_h = (y_{h1}, y_{h2}, \ldots, y_{hn_h})$ be a random sample of size n_h in stratum h from a Poisson distribution with parameter λ_h. The likelihood function based on the sample is

$$\ell\left(\lambda_h|y_h\right) = \frac{e^{-n_h\lambda_h}\lambda_h^{n_h\bar{y}_h}}{\prod_{i=1}^{n_h} y_{hi}!} \tag{4}$$

3 Classical Estimation of ω_h and λ_h

Following Sihm et al. (2016), the classical estimate $\hat{\omega}_h^c$ of ω_h is

$$\hat{\omega}_h^c = \frac{\hat{\theta}_h - (1 - p_{bh})\,\pi_b}{p_{bh}} \tag{5}$$

where $\hat{\theta}_h$ is the proportion of 'Yes' responses obtained in randomization device 1.
The classical estimate $\hat{\pi}_{hc}$ of π_h is

$$\hat{\pi}_{hc} = \frac{\hat{\phi}_h - (1 - p_{ah})\hat{\omega}_h^c \pi_{ah}}{1 - (1 - p_{ah})\hat{\omega}_h^c} \tag{6}$$

and $\hat{\phi}_h$ is the proportion of 'Yes' responses obtained in randomization device 2.

4 Bayesian Estimation of ω_h and λ_h

Using Bayes Theorem for random variables, we now update prior information in light of the sample information given by (2) and (4) to provide posterior information.

Result 1. *On assuming beta distribution as the prior distribution for ω_h with parameters u_h and v_h, $(u_h, v_h) > 0$,*

$$p\left(\omega_h\right) = \frac{1}{B\left(u_h, v_h\right)} \omega_h^{u_h - 1} \left(1 - \omega_h\right)^{v_h - 1}, 0 < \omega_h < 1$$

where $B(u_h, v_h) = \frac{\Gamma(u_h)\Gamma(v_h)}{\Gamma(u_h + v_h)}$ is the beta function.
The posterior distribution $p(\omega_h|y_h)$ of ω_h given Y_h is

$$p\left(\omega_h|y_h\right) = \frac{\sum_{i=1}^{y_h}\sum_{j=0}^{n_h - y_h}\binom{y_h}{i}\binom{n_h - y_h}{j} f_h^{y_h - i} k_h^{n_h - y_h - j} \omega_h^{i + u_h - 1}\left(1 - \omega_h\right)^{j + v_h - 1}}{\sum_{i=1}^{y_h}\sum_{j=0}^{n_h - y_h}\binom{y_h}{i}\binom{n_h - y_h}{j} f_h^{y_h - i} k_h^{n_h - y_h - j} B\left(i + u_h, j + v_h\right)}$$

Result 2. *Under the squared error loss function $L(\omega_h, \hat{\omega}_h) = (\omega_h - \hat{\omega}_h)^2$, the Bayes estimate $\hat{\omega}_h^s$ of the sensitivity level ω_h is posterior mean given by*

$$\hat{\omega}_h^s = E\left(\omega_h|y_h\right) = \frac{\sum_{i=1}^{y_h}\sum_{j=0}^{n_h - y_h}\binom{y_h}{i}\binom{n_h - y_h}{j} f_h^{y_h - i} k_h^{n_h - y_h - j} B\left(i + u_h + 1, j + v_h\right)}{\sum_{i=1}^{y_h}\sum_{j=0}^{n_h - y_h}\binom{y_h}{i}\binom{n_h - y_h}{j} f_h^{y_h - i} k_h^{n_h - y_h - j} B\left(i + u_h, j + v_h\right)} \tag{7}$$

The corresponding minimal value of Bayes posterior expected loss (BPEL) of $\hat{\omega}_h^s$ is

$$E(L(\omega_h, \hat{\omega}_h^s)|y_h) = E((\omega_h - \hat{\omega}_h^s)^2|y_h) = Var(\omega_h|y_h) = E(\omega_h^2|y_h) - E^2(\omega_h|y_h) \tag{8}$$

which is obtained by taking the expectation of the loss function $L(\omega_h, \hat{\omega}_h^s) = (\omega_h - \hat{\omega}_h^s)^2$ with respect to the posterior distribution $p(\omega_h|y_h)$. Bayesian posterior expected loss is a fundamental concept in Bayesian decision theory that measures the anticipated loss related to a decision, informed by the posterior distribution of parameters after data has been observed. The value of Bayesian posterior expected loss lies in its ability to improve decision-making by systematically managing uncertainty and optimizing outcomes based on updated beliefs. A minimal BPEL indicates that a specific decision is the most favourable choice considering the available data and prior beliefs, suggesting that, in the face of uncertainty, this decision is expected to result in the least loss or expense.

Remark 1. *Under non-informative prior, the Bayes estimate of ω_h is*

$$\hat{\omega}_h^* = \frac{\sum_{i=1}^{y_h}\sum_{j=0}^{n_h-y_h}\binom{y_h}{i}\binom{n_h-y_h}{j} f_h^{y_h-i}k_h^{n_h-y_h-j}B(i+2,j+1)}{\sum_{i=1}^{y_h}\sum_{j=0}^{n_h-y_h}\binom{y_h}{i}\binom{n_h-y_h}{j} f_h^{y_h-i}k_h^{n_h-y_h-j}B(i+1,j+1)}$$

which is obtained by putting $u_h = v_h = 1$.

Result 3. *Assume non-gamma prior for $\lambda_h > 0$, $p(\lambda_h) = K(\lambda_h)g(\lambda_h)$ where*

$$g(\lambda_h) = Gamma(\alpha_h, \beta_h), (\alpha_h, \beta_h > 0)$$

$$K(\lambda_h) = 1 + \frac{\delta_3\sqrt{\alpha_h}}{6(\alpha_h+1)(\alpha_h+2)}\left(L_3(\lambda_h) - \frac{3}{\alpha_h+3}L_4(\lambda_h)\right) + \frac{\delta_4\alpha_h}{24(\alpha_h+1)(\alpha_h+2)(\alpha_h+3)}L_4(\lambda_h)$$

and

$$L_r(\lambda_h) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\frac{\Gamma(\alpha_h+r)}{\Gamma(\alpha_h+r-i)}(\beta_h\lambda_h)^{r-i}, L_0(\lambda_h) = 1, r = 1,2,\ldots,m$$

The posterior distribution $p(\lambda_h|y_h)$ of λ_h given Y_h is given by

$$p(\lambda_h|y_h) = \frac{K(\lambda_h)}{G(\delta_3,\delta_4)}g(\lambda_h|y_h)$$

where

$$g(\lambda_h|y_h) = Gamma(\alpha'_h, \beta'_h) \text{ with } \alpha'_h = \alpha_h + n_h\bar{y}_h \text{ and } \beta'_h = \beta_h + n_h$$

and

$$G(\delta_3,\delta_4) = \int_0^\infty K(\lambda_h)g(\lambda_h|y_h)d\lambda_h = 1 - \delta_3\frac{\alpha^{\frac{3}{2}}}{6}C_1(\alpha'_h) + \delta_4\frac{\alpha^2}{24}C_2(\alpha'_h),$$

with
$C_1(\alpha'_h) = 3R_4 - 13R_3 + 21R_2 - 15R_1 + 4R_0,$
$C_2(\alpha'_h) = R_4 - 4R_3 + 6R_2 - 4R_1 + R_0,$
and $R_j = \frac{\mu_j}{\nu_j} = \left(\frac{\Gamma(\alpha'_h+j)}{\Gamma(\alpha'_h)\beta'^j_h}\right)\Big/\left(\frac{\Gamma(\alpha_h+j)}{\Gamma(\alpha_h)\beta^j_h}\right), j = 0,1,..,4$ *, where ν_j and μ_j are the moments of order j about the origin of $g(\lambda_h)$ and $g(\lambda_h|y_h)$, respectively.*

Proof. See Appendix A.1. □

Remark 2. *For $\delta_3 = \delta_4 = 0$, the posterior distribution $p(\lambda_h|y_h)$ reduces to Gamma Distribution with parameters (α'_h, β'_h).*

Result 4. *Under Linex loss function*

$$L\left(\lambda_h, \hat{\lambda}_h\right) = \exp\left(c\left(\hat{\lambda}_h - \lambda_h\right)\right) - c\left(\hat{\lambda}_h - \lambda_h\right) - 1, \tag{9}$$

the Bayes' estimate $\hat{\lambda}_h^L$ of the rare attribute λ_h is

$$\hat{\lambda}_h^L = -\frac{1}{c}logE\left(e^{-c\lambda_h}\Big|y_h\right) \tag{10}$$

where

$$E\left(e^{-c\lambda_h}\Big|y_h\right) = \left(1+\frac{c}{\beta'_h}\right)^{\alpha'_h}\left(\frac{1-\frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_3(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_4(\alpha'_h)}{1-\frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_2(\alpha'_h)}\right)$$

with
$C_3\left(\alpha'_h\right) = 3Q_4 - 13Q_3 + 21Q_2 - 15Q_1 + 4Q_0,$
$C_4\left(\alpha'_h\right) = Q_4 - 4Q_3 + 6Q_2 - 4Q_1 + Q_0,$
and $Q_j = \dfrac{\left(\frac{\Gamma(\alpha'_h+j)}{\Gamma(\alpha'_h)\beta'^j_h}\right)}{\left(\frac{\Gamma(\alpha_h+j)}{\Gamma(\alpha_h)\beta^j_h}\right)}\left(1+\frac{c}{\beta^j_h}\right)^{-j} = R_j\left(1+\frac{c}{\beta'_h}\right)^{-j}.$

The corresponding minimal value of BPEL of $\hat{\lambda}^L_h$ is obtained by taking the expectation of the loss function $L(\lambda_h, \hat{\lambda}^L_h)$ with respect to the posterior distribution $p(\lambda_h|y_h)$, that is,

$$E\left(L\left(\lambda_h, \hat{\lambda}^L_h\right)\Big|y_h\right) = E\left(\left(\exp\left(c\left(\hat{\lambda}^L_h - \lambda_h\right)\right) - c\left(\hat{\lambda}^L_h - \lambda_h\right) - 1\right)\Big|y_h\right) = c\left(E\left(\lambda_h|y_h\right) - \hat{\lambda}^L_h\right) \tag{11}$$

where

$$E\left(\lambda_h|y_h\right) = \frac{\alpha'_h}{\beta'_h}\left(\frac{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1\left(\alpha'_h+1\right) + \frac{\delta_4\alpha_h^2}{24}C_2\left(\alpha'_h+1\right)}{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1\left(\alpha'_h\right) + \frac{\delta_4\alpha_h^2}{24}C_2\left(\alpha'_h\right)}\right)$$

Proof. See Appendix A.2. and A.3. □

Result 5. *Using (3), (7) and (10), the Bayes estimate of rare sensitive attribute π_h, is*

$$\hat{\pi}_h = \frac{\hat{\phi}_h - \hat{\omega}^s_h(1-p_{ah})\pi_a}{1 - \hat{\omega}^s_h(1-p_{ah})\pi_a} \tag{12}$$

where $\hat{\phi}_h = \frac{\hat{\lambda}^L_h}{n_h}$.

Remark 3. *For $\delta_3 = \delta_4 = 0$, the Bayes' estimate $\hat{\lambda}^L_h$ of the rare attribute λ_h will be*

$$\hat{\lambda}^L_h = -\frac{1}{c}log\left(1+\frac{c}{\beta'_h}\right)^{\alpha'_h}$$

which is the Bayes estimate of the rare sensitive attribute λ_h using gamma prior under Linex loss function. Thus, the corresponding Bayes estimate of π_h can easily be obtained using Result 5.

Remark 4. *Under non-informative prior, the Bayes estimate of λ_h is*

$$\hat{\lambda}^*_h = -\frac{1}{c}log\left(1+\frac{c}{n_h}\right)^{n_h\bar{y}_h}$$

which is obtained by putting $\alpha_h = \beta_h = 0$.

Result 6. *For small values of $|u|$, the Linex loss function (9) is almost symmetric and not too different from the squared error loss function (SELF)*

$$L_s(\lambda_h, \hat{\lambda}_h) = \frac{c^2}{2}\left(\hat{\lambda}_h - \lambda_h\right)^2.$$

Under SELF, the Bayes' estimate $\hat{\lambda}^s_h$ of the rare attribute λ_h is given by

$$\hat{\lambda}^s_h = E(\lambda_h|y_h) = \frac{\alpha'_h}{\beta'_h}\left(\frac{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1(\alpha'_h+1) + \frac{\delta_4\alpha_h^2}{24}C_2(\alpha'_h+1)}{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_2(\alpha'_h)}\right) \tag{13}$$

and, hence, the Bayes estimate of rare sensitive attribute π_h, is

$$\hat{\pi}_h^s = \frac{\hat{\phi}_h^s - \hat{\omega}_h^s (1 - p_{ah}) \pi_a}{1 - \hat{\omega}_h^s (1 - p_{ah}) \pi_a}, \tag{14}$$

where $\hat{\phi}_h^s = \frac{\hat{\lambda}_h^s}{n_h}$.

Remark 5. *For $\delta_3 = \delta_4 = 0$, (13) will be reduced to*

$$\hat{\lambda}_h^s = \frac{\alpha_h'}{\beta_h'}$$

which is the Bayes estimate of the rare sensitive attribute λ_h using gamma prior under SELF. Thus, the corresponding Bayes estimate of π_h can easily be obtained using (12).

5 Comparison of Classical and Bayes Estimates using Relative Losses

In order to compare the classical estimates given in (5) and (6) with that of Bayes estimates given in (7) and (12), the relative losses are evaluated using BPEL of both the Bayes as well as the classical estimates. The minimal BPEL, under SELF, for the Bayes estimate of $\hat{\omega}_h^s$ is derived in (8). The BPEL of the classical estimate $\hat{\omega}_h^c$ is

$$E\left(L\left(\omega_h, \hat{\omega}_h^c\right) | y_h\right) = E\left(\left(\omega_h - \hat{\omega}_h^c\right)^2 \Big| y_h\right) = E\left(\omega_h^2 | y_h\right) + \left(\hat{\omega}_h^c\right)^2 - 2\hat{\omega}_h^c E\left(\omega_h | y_h\right) \tag{15}$$

From (8) and (15), the loss in using $\hat{\omega}_h^c$ relative to $\hat{\omega}_h^s$ is

$$RL_1 = \frac{E\left(L\left(\omega_h, \hat{\omega}_h^c\right) | y_h\right) - E\left(L\left(\omega_h, \hat{\omega}_h^s\right) | y_h\right)}{E\left(L\left(\omega_h, \hat{\omega}_h^s\right) | y_h\right)} = \frac{\left(\hat{\omega}_h^c - \hat{\omega}_h^s\right)^2}{Var\left(\omega_h | y_h\right)} \tag{16}$$

The minimal BPEL, under Linex loss function, for the Bayes estimate of $\hat{\lambda}_h^L$ is derived in (11). The BPEL of the classical estimate $\hat{\lambda}_h^c$ is

$$\begin{aligned} E\left(L\left(\lambda_h, \hat{\lambda}_h^c\right) \Big| y_h\right) &= E\left(\left(\exp\left(c\left(\hat{\lambda}_h^c - \lambda_h\right)\right) - c\left(\hat{\lambda}_h^c - \lambda_h\right) - 1\right) \Big| y_h\right) \\ &= \exp\left(c\hat{\lambda}_h^c\right) E\left(\exp\left(-c\lambda_h\right) | y_h\right) - c\hat{\lambda}_h^c + cE\left(\lambda_h | y_h\right) - 1 \end{aligned} \tag{17}$$

From (11) and (17), the loss in using $\hat{\lambda}_h^c$ relative to $\hat{\lambda}_h^L$ is

$$RL_2 = \frac{E\left(L\left(\lambda_h, \hat{\lambda}_h^c\right) \Big| y_h\right) - E\left(L\left(\lambda_h, \hat{\lambda}_h^L\right) \Big| y_h\right)}{E\left(\left(\lambda_h, \hat{\lambda}_h^L\right) \Big| y_h\right)} = \frac{\exp\left(c\left(\hat{\lambda}_h^c - \hat{\lambda}_h^L\right)\right) - c\left(\hat{\lambda}_h^c - \hat{\lambda}_h^L\right) - 1}{c\left(E\left(\lambda_h | y_h\right) - \hat{\lambda}_h^L\right)} \tag{18}$$

One may observe from (16) and (18) that relative losses are non-negative. Thus, loss in using classical estimates will always be more than that of Bayes estimates. Hence, the Bayes estimates of both the parameters perform better than the corresponding classical estimates.

6 Robustness Study

In order to examine sensitivity of Bayes estimator of the rare sensitive attribute π_h to a possible mis-specification of the shape parameter 'c' of the Linex loss function as the prior varies over a subclass of K-prior, we generate 1000 responses of randomization device 2 from $Poisson(\lambda_h)$ where λ_h is a randomly generated from $Gamma(1,1)$ and the number of strata $h = 4$. Also, 1000 responses of randomization device 1 is generated from $Bernoulli(\theta_h)$ where θ_h is randomly generated from $Beta(1,1)$. Under the subclass of K-prior, (δ_3, δ_4) is chosen as $(0,0), (0.15,2), (0.75,5)$ and $(1.05,8)$, whereas, the values of shape parameter c are taken from -1 to 1 with an increment of 0.25. Also, $\pi_a = 0.00004$ and $\pi_b = 0.25$ for each stratum. The following tables gives the value of the Bayes estimator $\hat{\pi}_h$ of rare sensitive attribute given in (12). All calculations are performed using Mathematica software.

	STRATUM 1 ($p_a = 0.2, p_b = 0.65$)				**STRATUM 2 ($p_a = 0.35, p_b = 0.5$)**			
$c \downarrow, (\delta_3, \delta_4) \rightarrow$	**(0,0)**	**(0.15,2)**	**(0.75,5)**	**(1.05,8)**	**(0,0)**	**(0.15,2)**	**(0.75,5)**	**(1.05,8)**
-1	0.00048364	0.00048366	0.00048406	0.00048418	0.00063557	0.00063558	0.0006358	0.00063585
-0.75	0.00048357	0.00004835	0.00048399	0.00048411	0.00063549	0.00063550	0.00063571	0.00063577
-0.50	0.00048350	0.00048352	0.00048393	0.00048405	0.00063541	0.00063542	0.00063563	0.00063569
-0.25	0.00048344	0.00048346	0.00048386	0.00048398	0.00063533	0.00063534	0.00063555	0.00063561
-0.01	0.00048337	0.00048339	0.00048380	0.00048392	0.00063525	0.00063527	0.00063548	0.00063553
0.01	0.00048337	0.00048339	0.00048379	0.00048391	0.00063524	0.00063526	0.00063547	0.00063553
0.25	0.00048330	0.00048332	0.00048373	0.00048385	0.00063517	0.00063518	0.00063539	0.00063545
0.50	0.00048324	0.00048326	0.00048366	0.00048378	0.00063509	0.00063510	0.00063531	0.00063537
0.75	0.00048317	0.00048319	0.00048359	0.00048371	0.00063501	0.00063502	0.00063523	0.00063529
1	0.00048310	0.00048312	0.00048353	0.00048365	0.00063493	0.00063494	0.00063515	0.00063521
	STRATUM 3 ($p_a = 0.5, p_b = 0.35$)				**STRATUM 4 ($p_a = 0.65, p_b = 0.2$)**			
$c \downarrow, (\delta_3, \delta_4) \rightarrow$	**(0,0)**	**(0.15,2)**	**(0.75,5)**	**(1.05,8)**	**(0,0)**	**(0.15,2)**	**(0.75,5)**	**(1.05,8)**
-1	0.00100524	0.00100525	0.00100527	0.00100528	0.00027505	0.00027506	0.00027525	0.00027530
-0.75	0.00100512	0.00100513	0.00100514	0.00100516	0.00027501	0.00027502	0.00027521	0.00027527
-0.50	0.00100499	0.001005	0.00100502	0.00100503	0.00027498	0.00027499	0.00027518	0.00027523
-0.25	0.00100486	0.00100484	0.00100489	0.00100491	0.00027494	0.00027495	0.00027515	0.00027520
-0.01	0.00100474	0.00100476	0.00100477	0.00100479	0.00027491	0.00027492	0.00027511	0.00027517
0.01	0.00100473	0.00100475	0.00100476	0.00100478	0.00027491	0.00027492	0.00027511	0.00027516
0.25	0.00100461	0.00100463	0.00100464	0.00100466	0.00027487	0.00027488	0.00027508	0.00027513
0.50	0.00100449	0.0010045	0.00100452	0.00100453	0.00027484	0.00027485	0.00027504	0.00027510
0.75	0.00100436	0.00100438	0.00100439	0.00100441	0.00027481	0.0002748	0.00027501	0.00027506
1	0.00100424	0.00100425	0.00100427	0.00100428	0.00027477	0.00027478	0.00027497	0.00027503

Table 1: Values of the Bayes Estimator $\hat{\pi}_h$ for different choices of c and (δ_3, δ_4)

The following observations are made:

1. The Bayes estimate $\hat{\pi}_h$ decreases, though insignificantly, as c increases from -1 to 1 for a particular choice of (δ_3, δ_4) in all strata. Hence, one may conclude that it is insensitive with respect to change in the value of c and thus $\hat{\pi}_h$ can be considered to be loss robust with respect to a possible misspecification of the shape parameter of the Linex loss function.

2. In each stratum, the Bayes estimate $\hat{\pi}_h$ does not differ much with respect to (δ_3, δ_4) for all choices of c. Hence, one may conclude that it is insensitive with respect to change in the value of (δ_3, δ_4) and thus $\hat{\pi}_h$ can be considered to be prior robust with respect to a possible misspecification in the parameters of K-prior.

7 Conclusion

Upon considering the Bayesian set up of the stratified extension of Sihm et al. (2016) binary model, it has been observed through relative losses that the Bayes estimates of the sensitivity level and the prevalence of rare sensitive attribute are more efficient than their classical counterparts. The primary focus of the study is to examine robustness of Bayes' estimates with respect to misspecification in the loss function as well as the prior distribution. The numerical illustration provided herein suggests that the Bayes estimate of the latter variable is loss robust with respect to the shape parameter of the Linex loss function as well as prior robust with respect to parameters of the K-prior. An enhanced version of the stratified randomized response model, featuring more flexible distributions, could be a promising avenue for future research on surveys related to sensitive topics.

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9 Declaration of Statement of Interest

The authors have no relevant financial or non-financial interests to disclose.

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A Appendix

A.1 Proof (Result 3)

It is known that $p\left(\lambda_h|y_h\right)=\frac{\ell(\lambda_h|y_h)p(\lambda_h)}{\int_0^\infty \ell(\lambda_h|y_h)p(\lambda_h)d\lambda_h}$. Using $\ell\left(\lambda_h|y_h\right)=\frac{e^{-n_h\lambda_h}\lambda_h^{n_h\bar{y}_h}}{\prod_{i=1}^{n_h} y_{hi}!}$, and

$$p\left(\lambda_h\right)=K\left(\lambda_h\right)g\left(\lambda_h\right)=\left[1+\frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\left(L_3\left(\lambda_h\right)-\frac{3}{\alpha_h+3}L_4\left(\lambda_h\right)\right)\right.$$
$$\left.+\frac{\delta_4\alpha_h}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}L_4\left(\lambda_h\right)\right]e^{-\beta_h\lambda_h}\frac{\lambda_h^{\alpha_h-1}\beta_h^{\alpha_h}}{\Gamma(\alpha_h)}$$

we get

$$\begin{aligned}p\left(\lambda_h|y_h\right)&=\frac{e^{-(n_h+\beta_h)\lambda_h}\lambda_h^{n_h\bar{y}_h+\alpha_h-1}K\left(\lambda_h\right)}{\int_0^\infty e^{-(n_h+\beta_h)\lambda_h}\lambda_h^{n_h\bar{y}_h+\alpha_h-1}K\left(\lambda_h\right)d\lambda_h}\\&=\frac{g\left(\lambda_h|y_h\right)K\left(\lambda_h\right)}{\int_0^\infty g\left(\lambda_h|y_h\right)K\left(\lambda_h\right)d\lambda_h}=\frac{g\left(\lambda_h|y_h\right)K\left(\lambda_h\right)}{G\left(\delta_3,\delta_4\right)}\end{aligned}$$

where

$$g\left(\lambda_h|y_h\right)=Gamma\left({\alpha'}_h,{\beta'}_h\right)\text{ with }{\alpha'}_h=\alpha_h+n_h\bar{y}_h\text{ and }{\beta'}_h=\beta_h+n_h$$

and

$$\begin{aligned}G\left(\delta_3,\delta_4\right)&=\int_0^\infty K\left(\lambda_h\right)g\left(\lambda_h|y_h\right)d\lambda_h\\&=\int_0^\infty\left[1+\frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\left(L_3\left(\lambda_h\right)-\frac{3}{\alpha_h+3}L_4\left(\lambda_h\right)\right)\right.\\&\qquad\left.+\frac{\delta_4\alpha_h}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}L_4\left(\lambda_h\right)\right]g\left(\lambda_h|y_h\right)d\lambda_h\end{aligned}$$

$$= 1 + \left(\frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\right)\left(\frac{\Gamma\left(\alpha_h+3\right)}{\Gamma\left(\alpha_h\right)}\right)\sum_{i=0}^{3}(-1)^i\binom{3}{i}R_{3-i} + \left(\frac{\delta_4\alpha_h}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}\right)$$
$$\frac{\Gamma\left(\alpha_h+4\right)}{\Gamma\left(\alpha_h\right)}\sum_{i=0}^{4}(-1)^i\binom{4}{i}R_{4-i}$$

$$= 1 + \left(\frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}\right)\left(R_3 - 3R_2 + 3R_1 - R_0\right) + \left(\frac{\delta_4\alpha_h^2 - 12\delta_3\alpha_h^{\frac{3}{2}}}{24}\right)\left(R_4 - 4R_3 + 6R_2 - 4R_1 + R_0\right)$$

$= 1 - \delta_3\frac{\alpha_h^{\frac{3}{2}}}{6}C_1\left(\alpha'_h\right) + \delta_4\frac{\alpha_h^2}{24}C_2\left(\alpha'_h\right)$,
with
$C_1\left(\alpha'_h\right) = 3R_4 - 13R_3 + 21R_2 - 15R_1 + 4R_0$,
$C_2\left(\alpha'_h\right) = R_4 - 4R_3 + 6R_2 - 4R_1 + R_0$,
and
$R_j = \frac{\mu_j}{\nu_j}\frac{=\left(\frac{\Gamma\left(\alpha'_h+j\right)}{\Gamma\left(\alpha'_h\right)\beta'^j_h}\right)}{\left(\frac{\Gamma\left(\alpha_h+j\right)}{\Gamma\left(\alpha_h\right)\beta^j_h}\right)}, j = 0, 1, .., 4$, where ν_j and μ_j are the moments of order j about the origin of $g\left(\lambda_h\right)$ and $g\left(\lambda_h|y_h\right)$, respectively.

A.2 Proof (Result 4)

$$E\left(e^{-c\lambda_h}\middle|y_h\right) = \frac{1}{G\left(\delta_3,\delta_4\right)}\int_0^\infty e^{-c\lambda_h}p\left(\lambda_h|y_h\right)d\lambda_h$$
$$= \frac{1}{G\left(\delta_3,\delta_4\right)}\int_0^\infty e^{-c\lambda_h}\left(1 + \frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\left(L_3\left(\lambda_h\right) - \frac{3}{\alpha_h+3}L_4\left(\lambda_h\right)\right)\right.$$
$$\left. + \frac{\delta_4\alpha_h}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}L_4\left(\lambda_h\right)\right)\left(\frac{\beta'^{\alpha'_h}_h}{\Gamma\left(\alpha'_h\right)}\lambda_h^{\alpha'_h-1}e^{-\beta'_h\lambda_h}\right)d\lambda_h$$

$$= \frac{1}{G\left(\delta_3,\delta_4\right)}\left(\frac{\beta'^{\alpha'_h}_h}{\Gamma\left(\alpha'_h\right)}\int_0^\infty \lambda_h^{\alpha'_h-1}e^{-\left(c+\beta'_h\right)\lambda_h}d\lambda_h\right.$$
$$+ \frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\sum_{i=0}^{3}(-1)^i\binom{3}{i}\frac{\Gamma\left(\alpha_h+3\right)}{\Gamma\left(\alpha_h+3-i\right)}\beta_h^{3-i}\frac{\beta'^{\alpha'_h}_h}{\Gamma\left(\alpha'_h\right)}\int_0^\infty \lambda_h^{\alpha'_h+3-i-1}e^{-\left(c+\beta'_h\right)\lambda_h}d\lambda_h$$
$$\left. + \frac{\delta_4\alpha_h - 12\delta_3\sqrt{\alpha_h}}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}\sum_{i=0}^{4}(-1)^i\binom{4}{i}\frac{\Gamma\left(\alpha_h+4\right)\beta_h^{4-i}\beta'^{\alpha'_h}_h}{\Gamma\left(\alpha_h+4-i\right)\Gamma\left(\alpha'_h\right)}\int_0^\infty \lambda_h^{\alpha'_h+4-i-1}e^{-\left(c+\beta'_h\right)\lambda_h}d\lambda_h\right)$$

$$= \frac{1}{G\left(\delta_3,\delta_4\right)}\left(\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h} + \frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\sum_{i=0}^{3}(-1)^i\binom{3}{i}\frac{\Gamma\left(\alpha_h+3\right)\beta_h^{3-i}\beta'^{\alpha'_h}_h\Gamma\left(\alpha'_h+3-i\right)}{\Gamma\left(\alpha_h+3-i\right)\left(c+\beta'_h\right)^{\alpha'_h+3-i}\Gamma\left(\alpha'_h\right)}\right.$$
$$\left. + \frac{\delta_4\alpha_h - 12\delta_3\sqrt{\alpha_h}}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}\sum_{i=0}^{4}(-1)^i\binom{4}{i}\frac{\Gamma\left(\alpha_h+4\right)}{\Gamma\left(\alpha_h+4-i\right)}\beta_h^{4-i}\frac{\beta'^{\alpha'_h}_h}{\left(c+\beta'_h\right)^{\alpha'_h+4-i}}\frac{\Gamma\left(\alpha'_h+4-i\right)}{\Gamma\left(\alpha'_h\right)}\right)$$
$$= \frac{1}{G\left(\delta_3,\delta_4\right)}\left(\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h} + \frac{\delta_3\sqrt{\alpha_h}}{6\left(\alpha_h+1\right)\left(\alpha_h+2\right)}\frac{\Gamma\left(\alpha'_h+3\right)}{\Gamma\left(\alpha'_h\right)}\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h}A_h\right.$$
$$\left. + \frac{\delta_4\alpha_h - 12\delta_3\sqrt{\alpha_h}}{24\left(\alpha_h+1\right)\left(\alpha_h+2\right)\left(\alpha_h+3\right)}\frac{\Gamma\left(\alpha'_h+4\right)}{\Gamma\left(\alpha'_h\right)}\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h}B_h\right)$$

where

$$A_h = \sum_{i=0}^{3} (-1)^i \binom{3}{i} \frac{\Gamma(\alpha_h)}{\Gamma(\alpha_h+3-i)} \beta_h^{3-i} \frac{\Gamma(\alpha'_h+3-i)}{(c+\beta'_h)^{3-i}\Gamma(\alpha'_h)}$$
$$= \frac{\Gamma(\alpha_h)}{\Gamma(\alpha_h+3)} \beta_h^{3} \frac{\Gamma(\alpha'_h+3)}{(c+\beta'_h)^{3}\Gamma(\alpha'_h)} - 3\frac{\Gamma(\alpha_h)}{\Gamma(\alpha_h+2)} \beta_h^{2} \frac{\Gamma(\alpha'_h+2)}{(c+\beta'_h)^{2}\Gamma(\alpha'_h)}$$
$$+ 3\frac{\Gamma(\alpha_h)}{\Gamma(\alpha_h+1)} \beta_h \frac{\Gamma(\alpha'_h+1)}{(c+\beta'_h)\Gamma(\alpha'_h)} - \frac{\Gamma(\alpha_h)}{\Gamma(\alpha_h)} \beta_h^{0} \frac{\Gamma(\alpha'_h)}{(c+\beta'_h)^{0}\Gamma(\alpha'_h)}$$
$$= Q_3 - 3Q_2 + 3Q_1 - Q_0$$

with $Q_j = \left(\frac{\frac{\Gamma(\alpha'_h+j)}{\Gamma(\alpha'_h)\beta'^{j}_h}}{\frac{\Gamma(\alpha_h+j)}{\Gamma(\alpha_h)\beta^{j}_h}}\right)\left(1+\frac{c}{\beta^{j}_h}\right)^{-j} = R_j\left(1+\frac{c}{\beta'_h}\right)^{-j}$.
Similarly
$B_h = Q_4 - 4Q_3 + 6Q_2 - 4Q_1 + Q_0$.

$$E\left(e^{-c\lambda_h}\middle| y_h\right) = \frac{1}{G(\delta_3,\delta_4)}\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h}\left(1 + \frac{\delta_3\sqrt{\alpha_h}\alpha_h(\alpha_h+1)(\alpha_h+2)}{6(\alpha_h+1)(\alpha_h+2)}(Q_3-3Q_2+3Q_1-Q_0)\right.$$
$$\left. + \frac{(\delta_4\alpha_h - 12\delta_3\sqrt{\alpha_h})\alpha_h(\alpha_h+1)(\alpha_h+2)(\alpha_h+3)}{24(\alpha_h+1)(\alpha_h+2)(\alpha_h+3)}(Q_4-4Q_3+6Q_2-4Q_1+Q_0)\right)$$
$$= \frac{1}{G(\delta_3,\delta_4)}\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h}\left(1 + \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}(-3Q_4+13Q_3-21Q_2+15Q_1-4Q_0)\right.$$
$$\left. + \frac{\delta_4\alpha_h^2}{24}(Q_4-4Q_3+6Q_2-4Q_1+Q_0)\right)$$
$$= \frac{1}{G(\delta_3,\delta_4)}\left(\frac{\beta'_h}{c+\beta'_h}\right)^{\alpha'_h}\left(1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_3(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_4(\alpha'_h)\right)$$

where
$C_3(\alpha'_h) = 3Q_4 - 13Q_3 + 21Q_2 - 15Q_1 + 4Q_0$ and $C_4(\alpha'_h) = Q_4 - 4Q_3 + 6Q_2 - 4Q_1 + Q_0$

$$E\left(e^{-c\lambda_h}\middle| y_h\right) = \left(1+\frac{c}{\beta'_h}\right)^{-\alpha'_h}\left(\frac{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_3(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_4(\alpha'_h)}{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_2(\alpha'_h)}\right)$$

A.3 Proof (Result 6)

$$E(\lambda_h|y_h) = \frac{1}{G(\delta_3,\delta_4)}\int_0^\infty \lambda_h\, g(\lambda_h|y_h)\, K(\lambda_h)\, d\lambda_h$$
$$= \frac{1}{G(\delta_3,\delta_4)}\frac{\alpha'_h}{\beta'_h}\int_0^\infty\left(\frac{\beta'^{\alpha'_h+1}_h}{\Gamma(\alpha'_h+1)}\lambda_h^{\alpha'_h+1-1}e^{-\beta'_h\lambda_h}\right)K(\lambda_h)\, d\lambda_h$$
$$= \frac{\alpha'_h}{\beta'_h}\left(\frac{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1(\alpha'_h+1) + \frac{\delta_4\alpha_h^2}{24}C_2(\alpha'_h+1)}{1 - \frac{\delta_3\alpha_h^{\frac{3}{2}}}{6}C_1(\alpha'_h) + \frac{\delta_4\alpha_h^2}{24}C_2(\alpha'_h)}\right)$$