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Study of Inventory Model for Perishable Items with Partial Backlogging and Permissible Delay in Payment

Ravendra Kumar[1], **Sushil Kumar**[2], **Manish Kumar**[2]
[1]Department of Mathematics, V R A L Govt Girls Degree College Bareilly, Bareilly, India
[2]Department of Mathematics and Astronomy, University of Lucknow , Lucknow, India
Corresponding author: Ravendra Kumar, *E-mail: rkpmbly@gmail.com*

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Abstract
It is common knowledge that one of the key issues with every inventory model is dealing with perishable goods. Under the premise of a lawful payment delay, this study presents an inventory model including changeable perishable costs & partial backlogging costs. The available research on the topic typically discusses condition where payment for an order is produced upon the inventory system's receipt of the product, & scarcities are either entirely backed up or entirely off course, presuming that the cost of backlogs is "inversely proportional" to "mark time" for upcoming recovery. The suggested model is presented & illustrated using numerical examples & sensitivity analysis.

Keywords: Inventory; partial backlog; allowable delay for perishable items; perishable items

1 Introduction

Perishables are foods that quickly spoil if left out of the refrigerator, most commonly eatable products. In the actual world, many things are perishable while they are being stored. Consequently, the management and preservation of inventory of perishable goods, such as dairy products, meat, fish, and poultry, as well as cooked foods, has become a top concern for decision-makers in the present.

The analysis of the inventory model for perishable item was started in 1960, & there have been published numerous research articles in this field. Result is, when deciding on the best "inventory policy" for the type of product, the harm brought by perishable not to be overlooked. One of the earliest "inventory models" for perishables having two periods was proposed by Van Zyl in 1964. Since there were numerous literature reviews on the inventory method at the time. The two most well-known academics, Ghare and Schrader (1963), developed numerous significant inventories models for degrading inventory goods of continuous demand. Philip & Covert (1973) made use of a fluctuating rate of decline of a two parameter "Weibull distribution" to create model for assumption of a constant rate order & without shortage. After that, Philip (1974) expanded the inventory model for perishable goods by incorporating, two-parameter "Weibull distribution's" variable rate of degrading , however all of the aforementioned models were only constructed for constant demand. For perishable goods, Jaiswal and Shah (1977) proposed an "order level inventory model" having a constant rate of degradation. Agrawal (1978) improved the model that Shah & Jaiswal's analysis from 1977 provided and gave an "inventory" system at the order level using which mistake is calculated & the ordinary inventory holding cost is decreased. Su et al. (1977) presented a system for production and inventories perishable items as an trade exponentially decaying demand for a specific time period . Mak, Hollier, & Park (1983) also propose a model with backlog constant rates in their paper. In some products, like custom goods. Calculating whether or not the backlog will be accepted was the key challenge.

On the other hand, this model frequently presupposes that the seller will be paid for the item as soon as they get the consignment. because a reasonable payment delay can benefit the seller financially. In order to optimise the seller's advantage and gain, the buyer may be given a fixed credit duration to satisfy the obligations. Recently, a number of scholars suggested an analytical inventory model that took into account the allowable payment delay. A single item inventory model is suggested by Goel (1985) on the assumption of a potential payment delay. As the examination of the best inventory management strategy in the face of trade credit for trade, Chung (1989) offered the discounted cash flow point of view. Later, Shinn et al. (1996) expanded Goel's model and took into account a quantity discount to lower the cost of shipping.

Recently, to concur with the current inventory condition in practise The model of inventory with constant rate of deterioration was developed by Jaggi et al. (1995) under the assumption of a legal payment delay. When the seller permits a payment delay,H wang et al. (1997) provided a model with a joint cost and lot dimensions,decision-making challenge for an exponential degrading product.

In reality, several clients would prefer to wait because of backlogs during that time of shortage, though some would not because of stray sales. The participant fills the consumer's need for the product. It might be regarded as the sales' profit loss. Therefore, it is important to factor in the chance cost associated with lost sales while modeling. Numerous studies make the assumption that the deficit is either totally lost or totally backlogged. There are some inventory systems for many stocks, including edible goods. Whether or not the backlog will be received depends largely on how long it will be before the next replenishment arrives. The backlog rate would be lower the longer the expected time As a result, the backlog rate varies and depends on when the upcoming replenishment is anticipated. This study's objective is to increase the application of an expanding calculative inventory model and to fit a more comprehensive inventory play up by assuming varying perishable rates and partial backlogging rates as well as the prerequisite for allowable delay in payment. This paper is organised as follows:

2 Notation

$I(t) =$ cost level of stocking up for a brief period of time "t".

$\alpha =$ "a scale parameter".

$\beta =$ "a shape parameter".

$f(t) =$ distribution of time perishable item.

$a =$ backlogging parameter which is positive constant.

$\mathrm{I}_y =$ Interest on item with annual rate.

$I_R =$ Interest charge which are invested (\$ year^{-1}).

$A =$ Order of price inventory (\$/order).

$C_0 =$ Cost of missed opportunity scale(\$ unit^{-1}).

$C_0[T_1, T] =$ Total applicable inventory Cost.

$C_0'[T_1, T] =$ Total applicable inventory Cost for $T_1 > D$.

$C_0''[T_1, T] =$ Total applicable inventory Cost for $T_1 \leq D$.

$h =$ Holding price without $''$interest charges$''$ (\$ unit^{-1}year^{-1}).

$D =$ Permisable delay in setting account.

$C' =$ Unit purchase cost.

$r =$ Yearly demand.

$S_c =$ Shortage Cost (\$ unit^{-1}year^{-1}).

$T =$ Duration of recovery period.

$T_1 =$ ″Time″ at which shortage start ″$0 \leq T_1 \leq T$″.

$U(T_1, T) =$ Average overall ″inventory cost″ (time/unit).

$U_1(T_1, T) =$ Average overall ″inventory cost″ (time/unit) for $T_1 > D$.

$U_2(T_1, T) =$ Average overall ″inventory cost″ (time/unit) for $T_1 \leq D$.

3 Assumptions

Here, we base our mathematical model creation on the following supposition.

1. The recuperation occurs infinitely quickly and instantly.
2. There is only one item in the inventory order.
3. The allocation of time perishable follows two limitation ″Weibull distribution″ $f(\frac{t}{\alpha}, \beta) = \frac{\alpha}{\beta^\alpha} t^{\alpha-1} \left\{ e^{-(\frac{t}{\beta})\alpha} \right\}$, where α, β are shape and scale parameter, correspondingly and $\alpha \geq 0, \beta \geq 0$.
4. During the period of shortages, shortages are acceptable; the rate of backlogs can change & depends on how long it takes for the next recovery. The backlog rate would be much lower the longer the wait time., When waiting times $T - t$ for impending recovery increase, the percentage of customers willing to consent backlogs as time t gets shorter. We defined backlogging rate be $\left[\frac{1}{1+\alpha(T-t)}\right]$ as being when inventory is a negative number in order to solve this problem. Let's assume that the backlog limitation is a constant that is positive $T_1 < t \leq T$.

   Prior to resolution of recovery account. Customer may use selling proceeds for obtain "interest" at a yearly rate $''I_y''$. However, the item that is still available after the constant credit period ends is proposed to be funded by "yearly rate" I_R where ″$I_R \geq I - y$″.

4 Mathematical Model

Here, we investigate two scenarios where the action of the inventory model anytime, t can be expressed as in figure (a).

Case: (I)- If $D < T_1$ we first make assumption that period of payment is longer than period during which the product has a positive stock of inventory, Taking into account the stock level at time t, then Inventory costs arise as a result of the impact of demand and perishable during the time frame $(0, T_1)$. Therefore, a differential equation can be used to determine the difference in "inventory level" in relation with time.

$$\frac{dI(T)}{dt} = \{-r - f(t)I(t)\}, 0 \leq t \leq T - 1 \tag{1}$$

This is due to boundary condition ″$I(T_1) = 0$.

Answer of (1) explain as

$$I(t) = \left\{ re^{-\alpha t^\beta \int_0^{T_1} e^{\alpha\beta^n d\vartheta}} \right\} = \left[r \sum_{n=0}^{\infty} \frac{(-\alpha t^\beta)}{n!} \left(\sum_{n=0}^{\infty} \frac{\alpha^n (T_1^{1+\beta n} - t_1^{1+\beta n})}{n!(1+\beta n)} \right) \right] \tag{2}$$

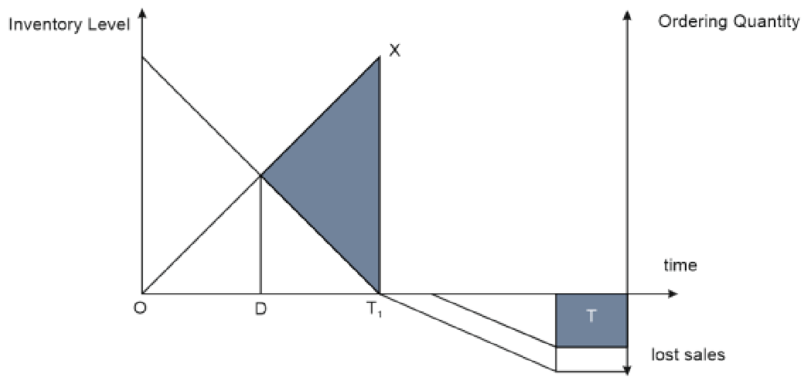

Figure 1: Mathematical model when $T_1 > D$

Where $0 \leq t \leq T_1$. Where T_1 is time at which shortage start.
By equation (2) holding cost "$I(t)$" where a little $''$time dt'' equivalent to $''hI(t)dt''$.
$\therefore$ Expense of keeping inventory over time $(0, T_1)$ is equal to $h\int_0^{T_1} I(t)dt$, in all, Consumer usage the Using sales revenue, calculate your profit margin annually. I_y in period $(0, T_1)$
then the profit earn is

$$C' I_y \int_0^{T_1} \gamma(T_1 - t)dt$$

and profit earn in annual rate I_R is

$$C' I_R \int_0^{T_1} I(t)dt$$

Let α is very little
Then, by disregarding the second-highest term, an approximation of the solution can be obtained of α, we will get

$$h\int_0^{T_1} I(t)dt \approx hr\left(\frac{T_1^2}{2} + \frac{\alpha\beta T_1^{2+h}}{(1+\beta)(2+\beta)}\right)$$

$$C' I_R \int_0^{T_1} I(t)dt \approx C' r I_R \left(\frac{D^2}{2} - \frac{\alpha\beta D^{2+\beta}}{(1+\beta)(2+\beta)}\right)$$
$$+ \frac{\alpha D^{1+\beta} T_1}{1+\beta} - DT_1 \frac{\left(1+\beta+\alpha T_1^{\beta}\right)}{1+\beta}$$
$$+ \frac{T_1}{2} + \frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}$$

Again, perishable cost per cycle is

$$C'\left(\gamma\int_0^{T_1} e^{\alpha t^{\beta}} dt - rT_1\right) \approx \frac{\alpha C' \gamma T_1^{1+\beta}}{1+\beta} \tag{3}$$

Additionally, inventory runs out as a result of the backlog in demand at the time(T_1, T).
$\therefore$ The differential equation below can be used to determine the difference in "inventory" with regard to time t.

$$\frac{dI(T)}{dt} = \left\{\frac{-r}{1+a(T-t)}\right\}, T_1 < t < T \tag{4}$$

By using "boundary condition" from $''I(T_1) = 0''$
Result of (3) becomes

$$I(T) = \left\{-\int_{t_1}^{T} \frac{r}{1+a(T-u)} du\right\}, T_1 < t < T \tag{5}$$

Now from equation (4) shortage expense over a little amount of time $-S_C I(T)dt$.
$\therefore$ Shortage cost over the period (T_1, T) becomes

$$S_C \int_{T_1}^{T} \int_{T_1}^{t} \frac{r}{1+a(T-u)} dudt = \frac{S_C}{a^2} r \left[a(T-T_1) - \log\left\{1+a(T-T_1)\right\}\right] \tag{6}$$

Next the good time cost of lost revenue during replenishment cycles is

$$C_0 \int_{T_1}^{T} r \left(1 - \frac{1}{1+a(T-t)}\right) dt = \frac{C_0}{a} r \left[\left\{a(T-T_1)\right\} - \log\left\{1+a(T-T_1)\right\}\right] \tag{7}$$

If the shortages are wholly backlogged Then $a = 0$
$\therefore$ Shortage cost will be $S_C r \frac{(T-T_1)^2}{2}$ and Sales that are lost during the replenishment cycle will have no opportunity cost.
Now that the overall cost of replenishment has been estimated, it is as follows:

$$C_0(T_1,T) = \frac{1}{T}\{\text{holding cost + perishable cost + order cost - profit gain + interest payable + shortage cost + opportunity cost due to lost sales}\}$$

$$\begin{aligned} C_0(T_1,T) = \Bigg[& \left[hr\left(\frac{T_1^2}{2} + \frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}\right)\right] + \left[\frac{\alpha C' r T_1^{1+\beta}}{1+\beta}\right] + \left[A - \frac{C' r I_y T_1^2}{2}\right] \\ & + C' r I_R \left(\frac{D^2}{2} - \frac{\alpha\beta D^{2+\beta}}{(1+\beta)(2+\beta)} + \frac{\alpha D^{1+\beta} T_1}{1+\beta} - D T_1 \frac{\left(1+\beta+\alpha T_1^{\beta}\right)}{1+\beta} + \frac{T_1^2}{2} \right. \\ & \left. + \frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}\right) + \frac{(S_C + C_0 a) r}{a^2}\left[a(T-T_1) - \log\left\{1+a(T-T_1)\right\}\right]\Bigg] \end{aligned} \tag{8}$$

The typical cost of all inventories $\frac{C_0'(T_1,T)}{T}$ a unit of time is indicated by $U_1(T_1,T)$, when shortage is not permitted $\Rightarrow'' T \to T_1''$, Thus cost of whole inventory on average per unit of time becomes

$$\begin{aligned} U_1(T_1) = \frac{1}{T_1}\Bigg[& A + hr\left(\frac{T_1^2}{2} + \frac{\alpha\beta T_1^{2+b}}{(1+\beta)(2+\beta)}\right) + \frac{\alpha C' r T_1^{1+\beta}}{1+\beta} - \frac{C' r I_y T_1^2}{2} \\ & + C' r I_R \left(\frac{D^2}{2} - \frac{\alpha\beta D^{2+\beta}}{(1+\beta)(2+\beta)} + \frac{\alpha D^{1+\beta} T_1}{1+\beta} - D T_1 \frac{\left(1+\beta+\alpha T_1^{\beta}\right)}{1+\beta} \right. \\ & \left. + \frac{T_1^2}{2} + \frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}\right)\Bigg] \end{aligned} \tag{9}$$

The ideal value is to reduce average overall inventory price per unit of time. $T_1 \to T_1^*$ & $T_2 \to T^*$ can be calculated through simultaneous resolution of the subsequent equation.

$$\left\{\frac{\delta U_1(T_1,T)}{\delta T_1}\right\} = 0$$

And

$$\left\{\frac{\delta U_1(T_1,T)}{\delta T}\right\} = 0$$

Which satisfy the condition

$$\left[\left(\frac{\delta^2 U_1(T_1,T)}{\delta T_1^2}\right)_{(T_1^* T^*)}\right] > 0, \left[\left(\frac{\delta^2 U_1(T_1,T)}{\delta T^2}\right)_{(T_1^* T^*)}\right] > 0,$$

and

$$\left[\left(\frac{\delta^2 U_1(T_1,T)}{\delta T_1^2}\right)\left(\frac{\delta^2 U_1(T_1,T)}{\delta T^2}\right)-\left(\frac{\delta^2 U_1(T_1,T)}{\delta T_1 \delta T}\right)\right]_{(T_1^* T^*)} > 0$$

The equation $\left\{\frac{\delta U_1(T_1,T)}{\delta T_1}\right\} = 0$ & $\left\{\frac{\delta U_1(T_1,T)}{\delta T}\right\} = 0$ are equivalent to (See Appendix A)
The equation $\left\{\frac{\delta U_1(T_1,T)}{\delta T_1}\right\} = 0$ & $\left\{\frac{\delta U_1(T_1,T)}{\delta T}\right\} = 0$ are comparable to

$$\left\{\frac{\delta U_1(T_1,T)}{\delta T_1}\right\} = \frac{1}{T}\left[hr\left(T_1+\frac{\alpha\beta T_1^{1+\beta}}{1+\beta}\right)+\alpha C' r T_1^{\beta}-\frac{r(S_C+C_0 a)(T-T_1)}{1+a(T-T_1)}-C' r I_y T_1 + \frac{C' r I_R}{1+\beta}\left(\alpha D^{1+\beta}-(1+\beta)D(1+\alpha T_1^{\beta})+T_1(1+\beta+2\beta T_1^{\beta})\right)\right]=0 \tag{10}$$

Also

$$\frac{\delta U_1(T_1,T)}{\delta T} = \left[\frac{r(S_C+C_0 a)(T-T_1)}{T(1+a(T-T_1))}\right]-\frac{1}{T^2}\left[A+hr\left(\frac{T_1^2}{2}+\frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}\right)+\frac{\alpha C' r T_1^{1+\beta}}{1+\beta} - \frac{C' r I_y}{2}\left(2DT_1-T_1^2\right)+\frac{(S_C+C_0 a)r}{a^2}\left[a(T-T_1)-\log\left\{1+a(T-T_1)\right\}\right] + C' r I_R\left(\frac{D^2}{2}-\frac{\alpha\beta D^{2+\beta}}{(1+\beta)(2+\beta)}+\frac{\alpha D^{1+\beta}T_1}{1+\beta}-DT_1\frac{\left(1+\beta+\alpha T_1^{\beta}\right)}{1+\beta} + \frac{T_1^2}{2}+\frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}\right)\right]=0 \tag{11}$$

For obtain “optimal values” of T_1 & T we minimize $''U_1(T_1,T)''$.

5 Solution Technique

Following solution technique has been provided in order to get the best solution.

5.1 Solution Technique 1

Step 1:

i. Commence with $T_1(1) = D$.

ii. Where $t_1(1)$ is time at which shortage start in first length of time.

iii. Calculating after entering the value of $T_1(1)$ into (9).

iv. By using the results of $T_1(1)$ calculating $T_1(2)$ from (10).

v. Continue using the same technique in (ii) & (iii) until there is without change in value of $\boldsymbol{a}$.

Step 2:

i. Compare D and T_1.

ii. If $T_1 > D, T_1$ is feasible then go step (3).

iii. descriptionIf $T_1 < D, T_1$ is not feasible then put $T_1 = D$ and figuring out the associated prices of T using equation (10) & move ahead in step (3).

Step 3:

i. Here is where you find the matching total average stock costs per unit of time $U_1(T_1^*, T^*)$.

Case: (II)- Assuming that the time period during $D \geq T_1$ where there is positive commodities stock levels is no longer than the credit period costs associated with maintaining inventory, perishable goods, shortages, and missed sales opportunities are all same to those in Case (I)

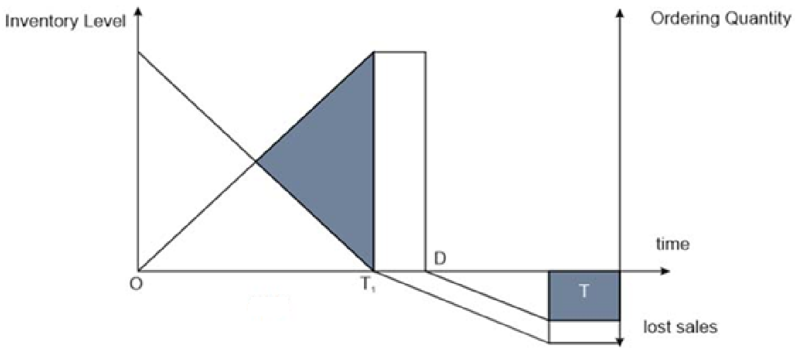

Figure 2: Mathematical model when $D \geq T_1$

$\because D \geq T_1$ Then the consumer without "interest & profit", A "interest" during period $[0, D]$ In this case profit earn is equal to

$$C'T_y \left[\int_0^{T_1} r(T_1 - t)dt - rT_1(D - T_1) \right]$$

$\therefore$ Total applicable "inventory cost" can be calculated as

$$C_0''(T_1, T) = [\text{holding cost + order cost + perishable cost - interest gained + shortage cost} \\ \text{+ opportunity cost}]$$

$$C_0''(T_1, T) = \left[hr \left(\frac{T_1^2}{2} + \frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)} \right) + A + \frac{\alpha C' r T_1^{1+\beta}}{1+\beta} - \frac{C' r I_y}{2}(2DT_1 - T_1^2) \right. \\ \left. + \frac{(S_C + C_0 a)r}{a_2} \left[a(T - T_1) - \log\{1 + a(T - T_1)\} \right] \right] \tag{12}$$

Therefore, the average cost per unit of all inventory $U_2(T_1, T)$ is equal to $\frac{C''_(T_-T)}{T}$ since Shortages are not permitted, so "average cost / unit" of all inventory will be

$$U_2(T_1) = \frac{1}{T_1} \left[A + hr \left(\frac{T_1^2}{2} + \frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)} \right) + \frac{\alpha C' r T_1^{1+\beta}}{1+\beta} - \frac{C' r I_y}{2}(2DT_1 - T_1^2) \right] \tag{13}$$

To reduce average cost of total inventory per unit of time. We can calculate T_1 and T by simultaneously resolving the subsequent equation. i.e.

$$\left\{ \frac{\delta U_2(T_1, T)}{\delta T_1} \right\} = 0$$

&

$$\left\{\frac{\delta U_2(T_1,T)}{\delta T}\right\}=0$$

and also satisfies the condition

$$\left[\left(\frac{\delta^2 U_2(T_1,T)}{\delta T_1^2}\right)_{(T_1^*,T^*)}\right]>0,\left[\left(\frac{\delta^2 U_2(T_1,T)}{\delta T^2}\right)_{(T_1^*,T^*)}\right]>0,$$

And

$$\left[\left(\frac{\delta^2 U_2(T_1,T)}{\delta T_1^2}\right)\left(\frac{\delta^2 U_2(T_1,T)}{\delta T^2}\right)-\left(\frac{\delta^2 U_2(T_1,T)}{\delta T_1\delta T}\right)^2\right]_{(T_1^*,T^*)}>0$$

See Appendix B

The equation $\frac{\delta U_2(T_1,T)}{\delta T_1}=0$ & $\frac{\delta U_2(T_1,T)}{\delta T}=0$ are equivalent to

$$\begin{aligned}\frac{\delta U_2(T_1,T)}{\delta T_1}=\frac{1}{T}&\left[hr\left(T_1+\frac{\alpha\beta T_1^{1+\beta}}{1+\beta}\right)+\alpha C_0 r T_1^{\beta}\right.\\&\left.-\frac{r(S_C+C_0a)(T-T_1)}{1+a(T-T_1)}-C_0 r I_y(D-T_1)\right]=0\end{aligned} \tag{14}$$

And

$$\begin{aligned}\frac{\delta U_2(T_1,T)}{\delta T}=&\left[\frac{r(S_C+C_0a)}{T\left(1+a(T-T_1)\right)}-\frac{1}{T^2}\left\{A+hr\left(\frac{T_1^2}{2}+\frac{\alpha\beta T_1^{2+\beta}}{(1+\beta)(2+\beta)}\right)\right.\right.\\&+\frac{aC'rT_1^{1+\beta}}{1+\beta}-\frac{C_0 r I_y}{2}(2DT_1-T_1^2)\\&\left.\left.+\frac{(S_C+C_0a)\gamma}{a_2}\left[a(T-T_1)-\log\left\{1+a(T-T_1)\right\}\right]\right\}=0\right]\end{aligned} \tag{15}$$

Here we explain following procedure to find the optimal (T_1,T).

5.2 Solution Technique 2

Step 1:

i. Start with $T_1(1)=D$.

ii. Putting value of $T_1(1)$ into (13) and then calculate $T_{(1)}$.

iii. With the help of $T_{(1)}$ calculate $T_1(2)$ from (14).

iv. Apply same procedure in (ii) and (iii) until the values of do not change of T_1 and T.

Step 2:

i. Compare D and T_1.

ii. If $T_1\leq D$ and T_1 is feasible then go Step (3).

iii. If $T_1>D$ and T_1 is not feasible then put $T_1=D$ and determining the corresponding T values from (14) then go step (3).

Step 3:

i. Here is where you find the matching total average stock costs per unit of time $U_2(T_1, T)$.

The objective of this issue is to identify “optimal values” of time T_1 & T_2 is minimum from conversation above we see that

$$U(T_1^*, T^*) = \text{min}\left[U_1(T_1^*, T^*), U_2(T_2^*, T^*)\right]$$

$\therefore$ Optimal order number is

$$r\left[\left\{\frac{\alpha}{1+\beta}\right\} + T_1^{*1+\beta} + T_1^* + \frac{1}{a}\log\left\{1 + a(T^* - T_1^*)\right\}\right]$$

6 Numerical Examples

For numerical illustration the proposed above model. We have examined following example.
Example 1-
Let $r = 1000$ unit/year, $a = 20.00$, $I_y = 0.14$ \$$^{-1}$/year, $A = \$\ 250.00$ per order, $I_R = 0.16$ \$$^{-1}$/year $C' = \$\ 100\,\text{unit}^{-1}$/year, $h = \$\ 15.00\,\text{unit}^{-1}$/year, $S_C = \$\ 40.00\,\text{unit}^{-1}$/year, $C_0' = 0.16\,\$\,\text{unit}^{-1}$/year $D = \frac{45}{365}$ year.

$f(t) = \alpha\beta t^{\beta-1}$, where $\alpha = 0.09, \beta = 1.6$.
After applying solution procedure 1 & 2, “optimal values” of $U_1(T_1, T) = 2335.700$ & $U_2(T_1, T) = 2505.550$.
Hence $U(T_1^*, T^*) = U_1(0.16540, 0.1733) = 2335.800$.
Then optimal order quantity is 174.00.

Exampe 2-
This kind of example. Considering that shortage is fully backlogged and also Suppose that variable is similar with example 1 “except $a = 0$”. “Optimal values” $U_1(T_1^*, T^*) = 2059.810$ & $U_2(T_1^*, T^*) = 2096.900$. Thus, optimal average overall “inventory cost / unit time”= 2059.800 & “optimal values” of T_1 & T will be $T_1^* = 0.15520$ & $T^* = 0.20670$ respectively.
$\therefore$ “Optimal order quantity” = 206.10.

Exampe 3-
In this type example, we suppose that shortage are not allowed and also suppose that variable are same as in example 1 but we not put $T = T_1$.
$\therefore$ The optimal of $U_1(T_1)$ and $U_2(T_2)$ will be $U_1(0.1670) = 2388.900$ and $U_1(0.11740) = 2593.900$.
Thus we get $U(T_1^*) = 2388.900$ and $T_1^* = 0.16730$ We obtain the ideal order quantity of 167.700.
Again, with the help of above example. Response to D-level changes and backlogging variable on “optimal values” of T_1, T & $U(T_1, T)$ being calculated.

7 Sensitivity Analysis

It is possible to calculate a “sensitivity analysis” by using “numerical illustration” above. Let different values of a are $1, 2.5, 5, 10, 25$. For each value of a we tested for three different values of D. The results are shown in table.

I. When a is fixed, D's value increases. As a result, the ideal average total cost will fall.

II. The optimal overall average cost will grow for fixed D as the value of a increases.

III. For fixed D as a increases in value. Without shortages, the optimal average total cost approaches the ideal total cost.

D	Complete Backlogging	$a = 1.00$	$a = 2.50$	$a = 5.00$	$a = 10.00$	$a = 25.00$	Without Shortage
30							
$U(T_1^*, T^*)$	2193.910	2277.970	2354.480	2424.930	2490.390	2531.700	2609.510
T_1^*	0.13800	0.14020	0.14310	0.14570	0.14820	0.15050	0.15260
T^*	0.19190	0.18340	0.17600	0.17600	0.16330	0.15770	0.15260
$\frac{T_1^*}{T^*}$	0.71430	0.76440	0.81320	0.86080	0.90770	0.95400	1.00000
45							
$U(T_1^*, T^*)$	2059.810	2128.580	2190.270	2246.260	2297.540	2344.890	2388.890
T_1^*	0.15320	0.15770	0.15100	0.16300	0.16400	0.16570	0.16730
T^*	0.20670	0.19800	0.19040	0.18370	0.17770	0.17230	0.16730
$\frac{T_1^*}{T^*}$	0.75080	0.79660	0.84030	0.88220	0.92270	0.96190	1.00000
60							
$U(T_1^*, T^*)$	1728.600	1796.050	1857.250	1913.420	1965.440	2013.100	2059.700
T_1^*	0.10850	0.11030	0.11190	0.11350	0.11490	0.11700	0.11740
T^*	0.15170	0.14500	0.13740	0.13160	0.12640	0.12170	0.11740
$\frac{T_1^*}{T^*}$	0.17500	0.76560	0.81450	0.86210	0.90870	0.95470	1.00000

Table 1: Effect of a and D on $U(T_1, T)$. Values of $U(T_1^*, T^*),\ T_1^*,\ T^*$or, $\frac{T_1^*}{T_2^*}$.

8 Conclusion

Within this article, we examine the inventory model for perishable goods with allowable payment delays and partial backlogs. In particular, we assume that the backlog rate decreases as the time before the next replenishment increases. Even though Goel (1985) did not consider the profit made during the cycle's remaining periods when the approval period is shorter than the cycle duration, this expectation is more reasonable in the market if we include and for the order with and once more. The models in each instance are identical to those from "Jamal et al." (1997), the use of two-variable "Weibull distribution" forces an expansion of application's scope. "Sensitivity analysis's" recommendations are likewise in line with financial motivation for fixing D the higher for the value of a.

The lower the "customer confirmation rates" that would be for backlogging at time t therefore, indirection to reduce the overall cost of inventory. If that an approaches infinity, the salesperson should indicate the portion of each cycle during which there are no shortages. At time t, the percentage of consumers willing to accept backlogs will be zero. Consequently, to reduce the overall cost of inventories. The marketer should divide by one before simulating a reduction in the absence of a shortage. Future studies may examine the results of using a demand rate that is more realistic.

9 Conflicts of Interest

"The authors declare no conflict of interest."

Appendix A

From equation (9) and (10) we get,

$$\left\{\frac{\delta U_1}{\delta T_1}(T_1,T)\right\}_{(T_1^*,T^*)}=0\ \&\ \left\{\frac{\delta U_1}{\delta T}(T_1,T)\right\}_{(T_1^*,T^*)}=0$$

Consider 2nd order "partial derivatives" of $U_1(T_1,T)$ w. r. to T_1 & T.

$$\left\{\frac{\delta^2 U_1(T_1,T)}{\delta T_1^2}\right\}_{(T_1^*,T^*)}=\frac{1}{T^*}\left[\alpha\beta C' r T_1^{*\beta-1}+hr(1+\alpha\beta T_1^{*\beta})+\frac{r(S_C+C_0a)}{T^*\left\{1+a(T^*-T_1^*)\right\}^2}\right.$$
$$\left.+\alpha\beta C' r I_R T_1^{*\beta-1}(T_1^*-D)+C' r(I_R-I_y)\right]>0.$$

$$\left[\left\{\frac{\delta^2 U_1(T_1,T)}{\delta T^2}\right\}_{(T_1^*,T^*)}\right]=\left[\frac{r(S_C+C_0a)}{T^*\left\{1+a(T^*-T_1^*)\right\}^2}\right]-\left[\frac{2}{T^*}\left\{\frac{\delta U_1(T_1,T)}{\delta T}\right\}_{(T_1^*,T^*)}\right]$$
$$=\frac{r(S_C+C_0a)}{T^*\left\{1+a(T^*-T_1^*)\right\}^2}>0$$

Now,

$$\left[\left\{\frac{\delta^2 U_1(T_1,T)}{\delta T_1\delta T}\right\}_{(T_1^*,T^*)}\right]=-\left[\frac{r(S_C+C_0a)}{T^*\left\{1+a(T^*-T_1^*)\right\}^2}\right]-\left[\frac{1}{T^*}\left\{\frac{\delta U_1(T_1,T)}{\delta T}\right\}_{(T_1^*,T^*)}\right]$$
$$=\frac{r(S_C+C_0a)}{T^*\left\{1+a(T_1^*-T^*)\right\}^2}$$

From above we also easily calculated that

$$\left[\left\{\frac{\delta^2 U_1(T_1,T)}{\delta T_1^2}\right\}_{(T_1^*,T^*)}\left\{\frac{\delta^2 U_1(T_1,T)}{\delta T^2}\right\}_{(T_1^*,T^*)}-\left\{\frac{\delta^2 U_1(T_1,T)}{\delta T_1\delta T}\right\}^2_{(T_1^*,T^*)}\right]$$

$$=r(S_C+C_0a)\frac{\left[\left\{h+C'(I_R-I_y)\right\}T_1^*+\alpha\beta\left\{hT_1^*+C'+C'(I_RT_1^*-I_yD)T^{*\beta}\right\}\right]}{T_1^*T^{*2}\left\{1+a(T^*-T_1^*)\right\}^2}>0$$

As a result, "Hessian matrix" at stationary point (T_1^*,T^*) is unambiguously positive in this condition. This is the least stationary position in our optimization issue, as we can see.

Appendix B

From equation (13) and (14)

$$\left\{\frac{\delta U_2}{\delta T_1}(T_1,T)\right\}_{(T_1^*,T^*)}=0 \ \& \left\{\frac{\delta U_2}{\delta T}(T_1,T)\right\}_{(T_1^*,T^*)}=0$$

Consider 2nd order "partial derivatives" of $U_2(T_1,T)$ w. r. to T_1 & T.

$$\left\{\frac{\delta^2 U_2(T_1,T)}{\delta T_1^2}\right\}_{(T_1^*,T^*)}=\left[\frac{1}{T^*}\left[\alpha\beta C' r T_1^{*\beta-1}+hr(1+\alpha\beta T_1^{*\beta})\right]\right.$$
$$\left.+\left\{\frac{r(S_C+C_0a)}{\{1+a(T^*-T_1^*)\}^2}+C' r I_y\right\}\right]>0.$$

$$\left\{\frac{\delta^2 U_2(T_1,T)}{\delta T^2}\right\}_{(T_1^*,T^*)}=\left[\frac{r(S_C+C_0a)}{T^*\{1+a(T^*-T_1^*)\}^2}\right]-\left[\frac{2}{T^*}\left\{\frac{\delta U_2(T_1,T)}{\delta T}\right\}_{(T_1^*,T^*)}\right]$$
$$=\left[\left\{\frac{r(S_C+C_0a)}{\{1+a(T^*-T_1^*)\}^2}\right\}\right]>0.$$

$$\left\{\frac{\delta^2 U_2(T_1,T)}{\delta T_1\delta T}\right\}_{(T_1^*,T^*)}=\left[\frac{r(S_C+C_0'a)}{T^*\{1+a(T^*-T_1^*)\}^2}\right]-\left[\frac{1}{T^*}\left\{\frac{\delta U_2(T_1,T)}{\delta T}\right\}_{(T_1^*,T^*)}\right]$$
$$-\left[\frac{r(S_C+C_0a)}{\{1+a(T^*-T_1^*)\}^2}\right]>0.$$

From above we also easily calculate that

$$\left[\left\{\frac{\delta^2 U_2(T_1,T)}{\delta T_1^2}\right\}_{(T_1^*,T^*)}\left\{\frac{\delta^2 U_2(T_1,T)}{\delta T^2}\right\}_{(T_1^*,T^*)}-\left\{\frac{\delta^2 U_2(T_1,T)}{\delta T_1\delta T}\right\}^2_{(T_1^*,T^*)}\right]$$
$$=\left[\frac{r(S_C+C_0a)\left[(h+C'I_y)T_1^*+\alpha\beta\left(hT_1^*+C'\right)T^{*\beta}\right]}{T_1^*T^{*2}\{1+a(T^*-T_1^*)\}^2}\right]>0$$

Thus, we can determine this stationary point (T_1^*,T^*) for our issue with optimization is least, since the "Hessian matrix" is positive definite in this case.

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