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Estimating Reliability in Multicomponent Stress-Strength Model: A Copula-Based Approach

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Abstract
In this study, the reliability of a multicomponent stress-strength model and its estimate are considered. When a component having strength X is exposed to a stress Y, its reliability is given by the probability $R = P(X > Y)$, where X and Y are both scalars. Extending the idea to a multicomponent situation, some authors take X and Y to be vectors. In the present study, we choose X to be a vector assuming the availability of several strengths while Y remains a scalar. The reliability under such a model is defined to be $R = P(\text{min strength} > Y)$. Two possible models for strength are considered, namely, the classical bivariate Clayton copula and Gumbel-Hougaard (GH) copula (Nelsen, 2006). The results obtained are evaluated graphically. Finally, a simulation study is conducted to have estimators of R.

Keywords: Stress-Strength Model, Copula, Reliability, Survival Function, Dependence Modeling

1 Introduction

In reliability analysis, the stress-strength (SS) model is used to describe the reliability of an component that has a random strength X operating in an environment that exerts random stress Y. The reliability of the component is then given by the probability that $R = P(X > Y)$. The idea behind this definition is that a component performs its intended task effectively as long as its strength exceeds the stress. Such situations arise, for example, in engineering, quality control, genetics, psychology, and so on. Interpretation of $P(X > Y)$ as an inequality measure between two income distributions has also come up in research.

The formal study of $P(X > Y)$ in its present form started with the seminal work of Birnbaum (1956) who connected the classical Mann-Whitney statistic with the SS model. Since then, researchers have extended the model in several directions. Thus, the conditions of normality and independence of X and Y have been overcome by assigning many other distributions to them and also allowing X and Y to be vectors. Many new and innovative definitions of reliability have been suggested depending on the nature of the problem.

Initially, X and Y were assumed to be univariate, independent, and normally distributed. Guo and Krishnamoorthy (2004) proposed some new approximate inferential methods for the reliability estimation assuming independent X and Y following normal distributions with unknown means and variances. Nandi and Aich (1994) obtained confidence bounds for $P(X > Y)$ assuming (X, Y) to follow a bivariate normal distribution.

The SS-model is also considered in the multicomponent set-up allowing X and Y to be vectors. Thus, Karadayi et al. (2011) proposed a model where a single strength X is subjected to two independent stresses Y_1 and Y_2 giving the reliability as $R = P(X > \max(Y_1, Y_2))$. Shawky et al. (2022), on the other hand, has studied an SS-model with $k(\geqslant 2)$ independent strengths and a single stress. They defined the reliability of their model as $R = P$ (at least s of the strengths $> Y$), $1 \leq s \leq k$.

Kunchur and Munoli (1993) have studied the multicomponent survival stress-strength model based on exponential distribution. Bai et al. (2018) have estimated the reliability of a multicomponent SS-model by assuming the dependent Weibull stress variables and exponential strength variables based on the Gumbel copula under the Type-I progressively hybrid censoring scheme.

The SS-model is also studied under a Bayesian perspective assigning prior distribution on the unknown parameter. For example, Nandi and Aich (1996) have conducted testing of the hypothesis for reliability under a restricted parameter space. However, much work remains to be done in this regard.

The use of copulas has become popular in all areas of dependence modeling. Strength and stress are not independent in reality, but dependent. However, their margins may be known while the joint distribution is unknown. In such situations, a suitably chosen copula may explain their dependence. Thus, Achcar et al. (2015) and Domma et al. (2012) took a copula-based approach to account for dependence in stress-strength models, along with others. In this study, the copula-based approach is followed.

There are gaps in research as mentioned in the above articles. For eg. Shawky et al. (2022) have taken the strengths to be independent. Overcoming the drawback we have assumed here correlated strengths following copula distributions, with the reliability given by $R = P(\min(\text{strength}) > Y)$.

This article is organized as follows. In section 2, we state the models of reliability which are considered in this article along with the assumptions made. Here, we prove a well-known conjecture of Nelsen connecting the Weibull distribution and the Gumbel-Hougaard copula. In section 3, the reliability is studied under the assumptions of the correlated strengths and a single stress. It is assumed that the two strengths jointly follow the bivariate Clayton copula. In section 4, an expression for reliability is obtained assuming a k-variate ($k \geq 2$) Gumbel-Hougaard copula for k strengths $X_1, X_2, ..., X_k$ and a single stress Y. In section 5, a simulation study is conducted to estimate reliability. Finally, section 6 gives concluding remarks.

2 Proposed Model and Conjecture of Nelsen

In this section, the assumed models for reliability have been discussed along with the assumptions made. Furthermore, a conjecture of Nelsen is stated and proved.

In the theory of probability, it is known that if the random variables $X_1, X_2, \cdots, X_k$ are independent, with the margins $F_1(x_1), F_2(x_2), \ldots, F_k(x_k)$, then the joint distribution can be obtained by multiplying the margins. If the X_i are not independent, then the joint distribution is obtained by Sklar theorem, as stated below.

Sklar's Theorem: Let H be a k-dimensional distribution function with margins $F_1(x_1), F_2(x_2), \ldots, F_k(x_k)$. Then, there exists a copula C such that

$$H(x_1, x_2, \ldots, x_k) = C(F_1(x_1), F_2(x_2), \cdots, F_k(x_k)). \tag{1}$$

If the F_i are continuous, then C is unique. Conversely, for any copula C and any $F_1, F_2, \cdots, F_k$, H defines a k-dimensional distribution function. We consider the following two models.

2.1 Model I: Bivariate Clayton Copula

The survival Clayton copula is given by

$$\overline{H}(x_1, x_2) = P\big(X_1 \geq x_1,\ X_2 \geq x_2\big) = C_L \tag{2}$$

where

$$C_L = \big(u_1^{-\frac{1}{\theta}} + u_2^{-\frac{1}{\theta}} - 1\big)^{-\theta}$$

(Clayton copula) with

$$u_1 = \overline{F}_1(x_1), \quad u_2 = \overline{F}_2(x_2), \quad 0 \leq u_1, u_2 \leq 1.$$

2.2 Model II: Gumbel Hougaard (GH) Copula

We first prove a conjecture of Nelsen that states that there is a one-to-one correspondence between the Weibull distribution and the Gumbel-Hougaard copula, which is a member of the Archimedean family.The conjecture is stated as a theorem and it's proof is given.

Theorem: Let $(X_1, X_2, \ldots, X_k)$ follow a multivariate Weibull distribution with the survival function

$$S(x_1, x_2, \ldots, x_k) = P(X_1 \geq x_1, X_2 \geq x_2, \ldots, X_k \geq x_k) = \exp\left\{-\left[\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \cdots + \left(\frac{x_k}{\lambda_k}\right)^{\frac{\gamma_k}{\alpha}}\right]^{\alpha}\right\} \tag{3}$$

where $0 \leq x_i < \infty$, $\quad 0 < \lambda_i < \infty$, $\quad 0 < \gamma_i < \infty$, and $0 < \alpha \leq 1$.
Then, (3) can be expressed in terms of the k-variate Gumbel-Hougaard copula, which is given by

$$\hat{C}_{GH}(\overline{u}) = \exp\left\{-\left[(-\ln u_1)^{\theta} + \cdots + (-\ln u_k)^{\theta}\right]^{\frac{1}{\theta}}\right\}, \tag{4}$$

where $\underline{u} = (u_1, u_2, \ldots, u_k)$ and $\hat{C}_{GH}$ is the survival copula.

Proof: From (3), the marginal survival function of X_1 is given by

$$S(x_1) = P(X_1 \geq x_1, X_2 \geq 0, \ldots, X_k \geq 0) = \exp\left\{-\left(\frac{x_1}{\lambda_1}\right)^{\gamma_1}\right\}. \tag{5}$$

$$\implies P(X_1 \geq x_1) = \exp\left\{-\left(\frac{x_1}{\lambda_1}\right)^{\gamma_1}\right\} = \overline{F}_{X_1}(x_1)$$

Putting $\overline{F}_1(x_1) = u_1$, we get

$$\left(\frac{x_1}{\lambda_1}\right)^{\gamma_1} = -\ln u_1.$$

Similarly,

$$\left(\frac{x_2}{\lambda_2}\right)^{\gamma_2} = -\ln u_2, \quad \ldots, \quad \left(\frac{x_k}{\lambda_k}\right)^{\gamma_k} = -\ln u_k.$$

So, from (3), we have

$$P(X_1 \geq x_1, X_2 \geq x_2, \ldots, X_k \geq x_k) = \exp\left\{-\left[(-\ln u_1)^{\theta} + \ldots + (-\ln u_k)^{\theta}\right]^{\frac{1}{\theta}}\right\}, \tag{6}$$

where $\theta = \frac{1}{\alpha}$. This agrees with the Gumbel-Hougaard Copula given in (4). This proves the conjecture.

Particular Case: Bivariate Gumbel-Hougaard Copula.
We put $k = 2$ in (6) to get the survival bivariate Gumbel-Hougaard Copula as

$$\hat{C}_{GH}(u_1, u_2) = \exp\left\{-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{\frac{1}{\theta}}\right\}, \tag{7}$$

where $1 \leq \theta < \infty, \quad and \quad 0 < u_1, u_2 < 1$.

3 The Case of Two Correlated Strengths and a Single Stress

Let us consider a system whose strength X is a vector (X_1, X_2) while stress Y is a scalar. Also, (X_1, X_2) and Y are independent. It is assumed that the system works if the minimum of the strengths exceeds the stress. This extends a model of Karadayi et al. (2011) which works if the single strength X exceeds the maximum of two stresses Y_1 and Y_2. The reliability in Karadayi's model is given as

$$R = P\left(X > \max\left(Y_1, Y_2\right)\right). \tag{8}$$

In our proposed model, the reliability is given by

$$R = P\left(\min\left(X_1, X_2\right) > Y\right), \tag{9}$$

where (X_1, X_2) follow the bivariate Clayton copula (see (2)) and Y follows an exponential distribution with mean β, and the margins of X_1 and X_2 are exponential with mean θ_0.

3.1 Calculation of the reliability R: Model I

We have

$$\begin{aligned} R &= P\left(\min(X_1, X_2) > Y\right) \\ &= \int_0^\infty P\left(X_1 > y, X_2 > y \mid Y = y\right) dG(y), \quad (G(y) \text{ being the distribution of } Y) \\ &= \int_0^\infty P\left(X_1 > y, X_2 > y\right) dG(y), \quad (\text{since } (X_1, X_2) \text{ is independent of } Y) \\ &= \int_0^\infty \overline{H}(y, y) dG(y). \quad (\text{using (2)}) \end{aligned} \tag{10}$$

On further simplification, we have

$$R = \int_0^\infty \left[\left(2\exp\left(\frac{y}{\theta\theta_0}\right) - 1\right)\right]^{-\theta} dG(y), \tag{11}$$

where $\theta, \theta_0 > 0$.

4 The Case of $k(\geq 2)$ Correlated Strengths and a Single Stress

Here, we consider a system where the strength X is a vector $(X_1, X_2, \cdots, X_k)$ while the stress Y is a scalar. Furthermore, $(X_1, X_2, \ldots, X_k)$ and Y are independent. The reliability of the system is given by

$$R = P\left(\min\left(X_1, X_2, \cdots, X_k\right) > Y\right), \tag{12}$$

where $(X_1, X_2, \cdots, X_k)$ follows a k-dimensional Gumbel-Hougaard Copula. This is a totally new approach to the study of reliability. Thus, as in the case of Shawky et al. (2022), we assume that all strengths are exposed to a common stress Y. But, unlike the idea of those authors, our systems will perform its task if the minimum of the strengths exceeds Y. However, the strengths are taken to be correlated. As per our knowledge, this is a new direction in evaluating the reliability of a multicomponent system.

4.1 Calculation of the Reliability: Model II

We have from (12),

$$
\begin{aligned}
R &= P(X_1 > Y, X_2 > Y, \ldots, X_k > Y) \\
&= \int_0^\infty P(X_1 > y, X_2 > y, \ldots, X_k > y \mid Y = y) dG(y), \quad (G(y) \text{ being the distribution of } Y) \\
&= \int_0^\infty P(X_1 > y, X_2 > y, \ldots, X_k > y) dG(y), \quad (\text{since } (X_1, X_2, \ldots, X_k) \text{ are independent of } Y)
\end{aligned}
$$

Choosing $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 1; \quad \gamma_1 = \gamma_2 = \ldots = \gamma_k = 1; \quad \alpha = \frac{1}{\theta}$, we have from (3) and above,

$$
\begin{aligned}
&= \int_0^\infty \exp\left\{ -\left(\left(\frac{y}{\lambda_1}\right)^{\theta\gamma_1} + \left(\frac{y}{\lambda_2}\right)^{\theta\gamma_2} + \cdots + \left(\frac{y}{\lambda_k}\right)^{\theta\gamma_k} \right)^{\frac{1}{\theta}} \right\} dG(y) \\
&= \int_0^\infty \exp\left\{ -\left(ky^\theta\right)^{\frac{1}{\theta}} \right\} \frac{1}{\beta} \exp\left\{ -\frac{y}{\beta} \right\} dy \quad (\text{since } G(y) \text{ is exponential with mean } \beta) \\
&= \frac{1}{\beta} \int_0^\infty \exp\left\{ -\left(k^{\frac{1}{\theta}} + \frac{1}{\beta} \right) y \right\} dy \\
&= \frac{1}{1 + \beta k^{\frac{1}{\theta}}}, \text{ where } \theta \geq 1, \quad \beta > 0, \quad \text{and} \quad k \geq 2 \qquad (13)
\end{aligned}
$$

The following Figures 1, 2, and 3 give the values of R for variation in θ for chosen values of β and k.

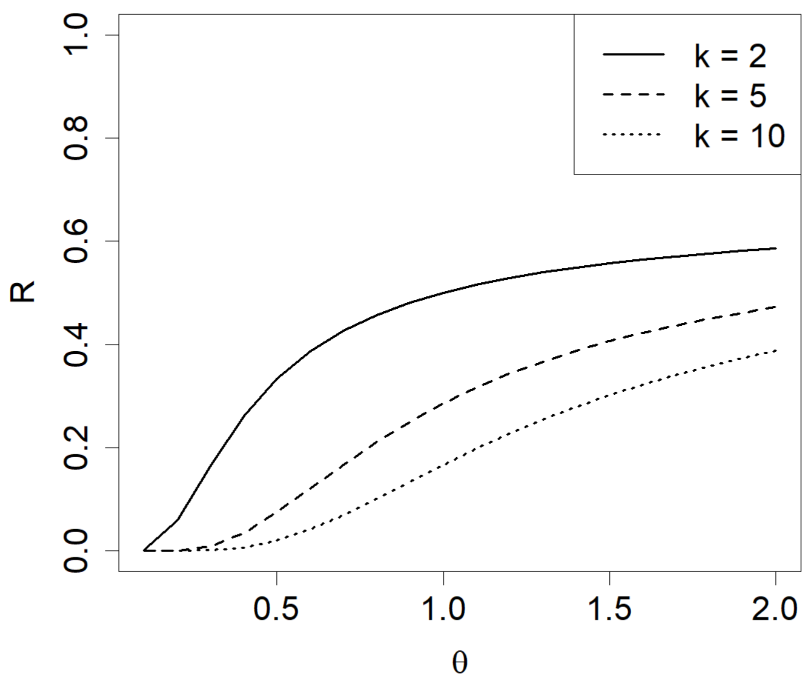

Figure 1: Graph of θ vs R for $\beta = 0.5$

From these graphs, it is noted that for each chosen value of β, the reliability R increases with the increase of θ (at each fixed value k). However, R decreases as k increases, for fixed β and θ.

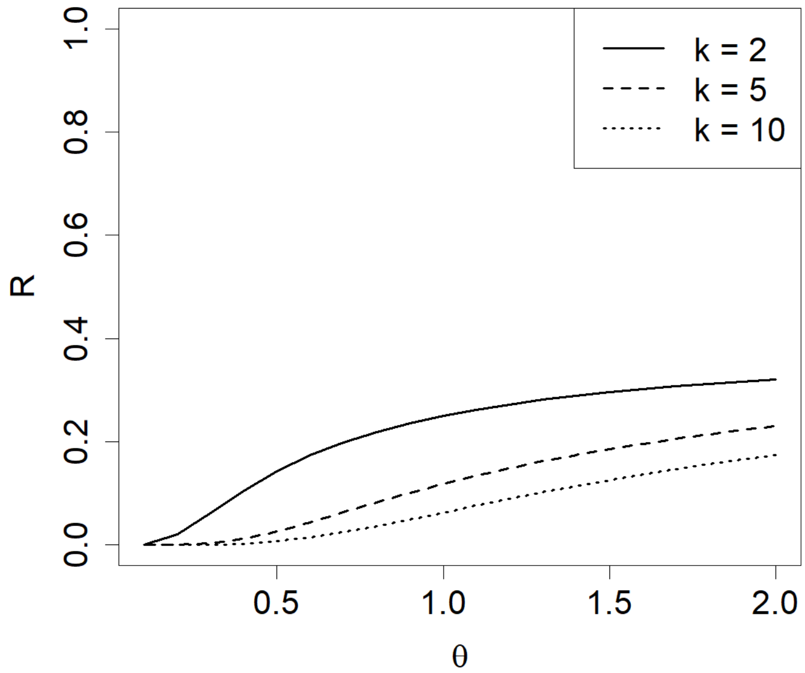

Figure 2: Graph of θ vs R for $\beta = 1.5$

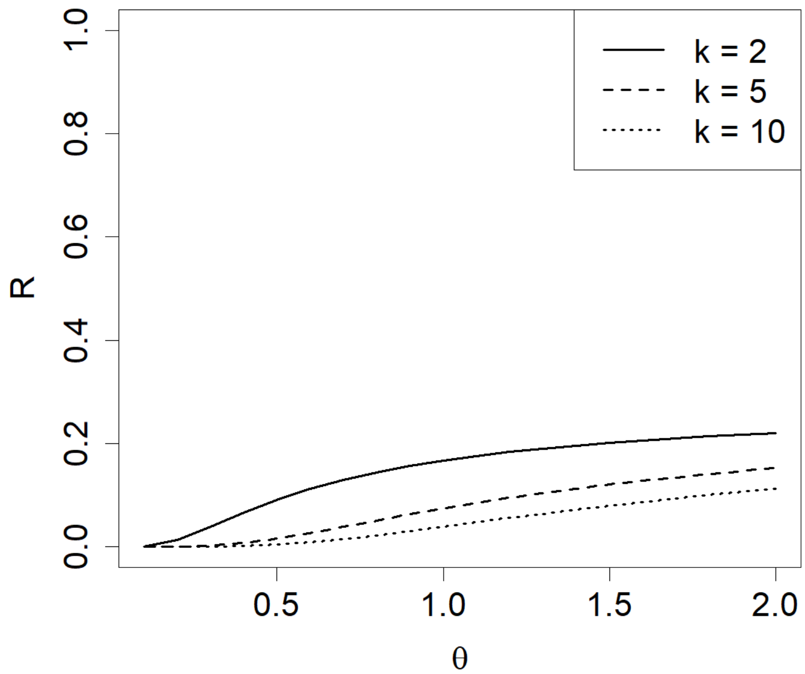

Figure 3: Graph of θ vs R for $\beta = 2.5$

No graph is possible in case of Model 1 as the reliability R cannot be written in a simple (i.e., closed) form. Also, we have taken fixed values of the parameters while simulating. This is done to keep the treatment and results simple.

5 Simulation Study to Estimate R

5.1 Model I

Under Model I, we have from (11)

$$R = \int_0^\infty a(y)dG(y). \tag{14}$$

Here,

$$a(y) = \left[\left(2\exp\left(\frac{y}{\theta\theta_0}\right) - 1\right)\right]^{-\theta} \tag{15}$$

where $y > 0$, $\theta, \theta_0 > 0$.

We further note that:

$$R = E(a(Y)) \tag{16}$$

Also, the CDF of Y, namely, $G(y)$, follows a $U(0,1)$ distribution. This gives:

$$Y = -\theta \ln U \tag{17}$$

where U is a random variable from $U(0,1)$.

5.1.1 Simulation Process

The simulation process can now be initiated by fixing $\theta = 1.5$, $\theta_0 = 2.0$, and $\beta = 2.5$. The following are the steps:

- Choose n independent U(0,1) random variables.
- Calculate $Y_1, Y_2, \ldots, Y_n$ from (17)
- Then, obtain $a(Y_1), a(Y_2), \ldots, a(Y_n)$ from equation (15).
- Finally,

$$\hat{R} = \frac{1}{n}\sum_{j=1}^{n} a(Y_j) \Rightarrow \hat{R_n}(say). \tag{18}$$

By the Strong Law of Large Numbers (SLLN), $\hat{R_n}$ converges to R for large n.

Sample Size n	$\hat{R_n}$
50	0.2924
100	0.3257
500	0.3628
1000	0.3316
5000	0.3227

Table 1: Estimated Reliability, assuming $\theta = 1.5$, $\theta_0 = 2.0$, $\beta = 2.5$

By SLLN $R \approx 0.3227$

5.2 Model II

Under Model II, we have the reliability R as:

$$R = \frac{1}{1 + \beta k^{\frac{1}{\theta}}} \tag{19}$$

where $\theta > 1$, $\beta > 0$, and $k \geq 2$.

For fixed parameter values, the reliability R is calculated and shown in the following tables.

k	R
1.0	0.2857
1.5	0.2339
2.0	0.2013
2.5	0.1784
5.0	0.1203

Table 2: Estimated Reliability for $\theta = 1.5$, $\beta = 2.5$

θ	R
1.0	0.1667
1.5	0.2013
2.0	0.2205
2.5	0.2326
5.0	0.2583

Table 3: Estimated Reliability for $k = 2$, $\beta = 2.5$

β	R
1.0	0.2548
1.5	0.1857
2.5	0.1203
3.0	0.1023
3.5	0.0890

Table 4: Estimated Reliability for $k = 5$, $\theta = 1.5$

6 Concluding Remarks

In this paper, we have proposed two new definitions of reliability in the context of a multicomponent stress-strength model. Assuming the availability of k strengths $X_1, X_2, \ldots, X_k$ and single stress Y

which is independent of strengths, the proposed definitions of reliability R are:

1. (a) $R = P(\min(X_1, X_2) > Y)$ in the bivariate case, and
2. (b) $R = P(\min(X_1, X_2, \ldots, X_k) > Y)$ in the k-variate case.

In case (a), we use the Bivariate Clayton copula, and in case (b), we use the k-variate Gumbel-Hougaard Copula for strengths. The distribution of Y is assumed to be exponential in both the cases.

The ideas presented in this paper can be extended in several directions. For instance, let $Y = (Y_1, Y_2, \ldots, Y_l)$ denote a vector comprising l stress components. In this case, the reliability $\mathcal{R}$ may be defined as

$$\mathcal{R} = P(\min(X_1, X_2, \ldots, X_k) > \max(Y_1, Y_2, \ldots, Y_l)).$$

Additionally, other copula families may be explored to improve the fit to real-world data. However, incorporating these modifications into the inference procedure could lead to increased computational complexity and analytical challenges.

In this study, simulation is conducted for fixed parameter values. However, it is possible to simulate observations from the Clayton $\&$ Gumbel-Hougaard copulas as well as mutually independent exponential distributions, and estimate θ, θ_0, and β, and then obtain an estimate of R. This real but difficult case maybe taken up for future research.

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8 Conflicts of Interest

The authors declare no conflict of interest.

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