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On random Fourier–Hermite Transform Associated with Stochastic Process

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Abstract

Liu and Liu in 2007 introduced the random Fourier–Hermite transform $\sum \tilde{a}_n \lambda_n^{\tilde{R}} \psi_n(t)$ which is a random Fourier - Hermite series with random coefficients $\lambda_n^{\tilde{R}}$ chosen randomly from the unit circle in $\mathbb{C}$, where $\psi_n(t)$ are Hermite functions and $\tilde{a}_n$ are Fourier–Hermite coefficients of an $L^2(\mathbb{R})$ function. They used it in image encryption and decryption and expected its application in general signal and image processing. This motivated us to investigate more on random Fourier–Hermite transform, replacing the random variables $\lambda_n^{\tilde{R}}$ by other random variables. It leads to address two problems. First, to focus on the convergence of random Fourier–Hermite series. Secondly, to find out the Fourier transform of the sum function of these random Fourier–Hermite series. The random variables those has been chosen are Fourier–Hermite coefficients of a stochastic process. They are found to be independent, if associated with the Wiener process and a symmetric stable process of index 2. We could establish convergence of the random Fourier–Hermite series if the scalars $\tilde{a}_n$ are Fourier - Hermite coefficients of functions of an $L^2(\mathbb{R})$ function. The Fourier transform of generalized Hermite functions and transformed Hermite functions are found out which aid in finding the Fourier transform of the sum functions of these random Fourier–Hermite series.

Keywords: Symmetric stable process, Stochastic integral, Random Fourier-Hermite series, Convergence in mean, Convergence in probability.

1 Introduction

Let $W(\delta, \omega)$, be the Wiener process for $\delta \in \mathbb{R}$ and ω in the sample space Ω. Hunt (1951) could define stochastic integral $\int_{-\infty}^{\infty} f(\delta) W(d\delta, \omega)$ for $f \in L^2(\mathbb{R})$ as a limit of Riemann–Stieltjes sum which is a random variable. This led him to define random Fourier transform

$$\int_{-\infty}^{\infty} e^{i\delta t} f(\delta) dW(\delta, \omega)$$

for $f \in L^2(\mathbb{R})$. Liu & Liu (2007) introduced random Fourier transform in Hermite polynomials. It is known that, $f \in L^2(\mathbb{R})$ has the Fourier–Hermite series expansion

$$\sum_{n=0}^{\infty} \tilde{a}_n \psi_n(t) \tag{1}$$

where, $\tilde{a}_n$ are the Fourier–Hermite coefficients of f [see Muckenhoupt (1970); Wiener (1933)]. Fourier transform of f, denoted as $\mathcal{F}(f)$ is found to be

$$\mathcal{F}(f(t)) = \sum_{n=0}^{\infty} \tilde{a}_n \lambda_n \psi_n(t), \tag{2}$$

using the fact that

$$\mathcal{F}(\psi_n(t)) := \lambda_n \psi_n(t), \qquad t \in \mathbb{R} \tag{3}$$

where, $\lambda_n = e^{\frac{-in\pi}{2}}$ takes only 4 possible values $\{1, -1, i, -i\}$[cf. Askey & Wainger (1965); Liu & Liu (2007)]. Since the fractional Fourier transform of order β, for $\beta \in \mathbb{Q}$ is

$$\mathcal{F}^{\beta}(\psi_n(t)) := \lambda_n^{\beta} \psi_n(t), \tag{4}$$

it is true that

$$\mathcal{F}^{\beta}[f(t)] := \sum_{n=0}^{\infty} \tilde{a}_n \lambda_n^{\beta} \psi_n(t). \tag{5}$$

Liu & Liu (2007) extended Fourier transform(FT) of irrational order and obtained the series

$$\mathcal{F}^{\tilde{R}}(f(t)) := \sum_{n=0}^{\infty} \tilde{a}_n \lambda_n^{\tilde{R}} \psi_n(t) \tag{6}$$

where, $\lambda_n^{\tilde{R}}$ are randomly chosen scalars from the unit circle in $\mathbb{C}$. He introduced the series (6) as the random Fourier transform (RFT), which is a random Fourier series(RFS) in Hermite polynomials. They used it in image encryption and decryption. They anticipate that this RFT approach will be necessary for general signal and image processing and digital and optical image encryption. This work opens a path to explore RFT. Findings in this direction may be helpful in physical sciences in the future.

From the point of Mathematical curiosity, it is interesting to know what will happen if any other random variables replace the random variables of absolute value one. This replacement will give rise to random Fourier–Hermite Series(RFHS),

$$\sum_{n=0}^{\infty} \tilde{a}_n X_n \psi_n(t), \tag{7}$$

where X_n are some random variables.

Initially RFS

$$\sum_{n \in \mathbb{Z}} a_n X_n e^{int} \tag{8}$$

arose in the context of white noise, where X_n are independent identically distributed random variables, and a_n are Fourier coefficients of a function. The model for white noise was taken to be $\frac{dW}{dt}$ where W is a Brownian motion. Subsequently, stable processes were found to be better models for white noise than the Wiener process. Literatures on RFS involving stable processes are found in [cf. Dash & Pattanayak (2008); Nayak, Pattanayak & Mishra (1987); Pattanayak & Sahoo (2005)].

In this work, the random variables X_n are chosen as the Fourier coefficients of a continuous stochastic process like the Wiener and stable processes. While dealing with the Wiener process, the random coefficients X_n are found to be independent but remain no longer independent if the stochastic process is stable process of index not equal to 2. A natural question arises about the convergence of the random series (7). The works on RFS associated with stochastic processes gives a way to look into these problems.

It is well known that $\int_a^b f(s)dW(s,\omega)$ is defined in quadratic mean if $f \in L^2[a,b]$ and is a random variable, where $W(s,\omega)$ is a Wiener process. Additionally, the stochastic integral

$$\int_a^b f(s)dX(s,\omega), \tag{9}$$

is defined in the sense of probability and is a random variable if $X(s,\omega)$, $s \in \mathbb{R}$ is a continuous stochastic process with independent increments, and f is a continuous function in $[a,b]$[cf. Lukacs (1975)]. If

$X(s,\omega)$ is a stochastic process that is symmetrically stable of index $\gamma \in (1,2]$ and has an independent increment, then the integral (9) is defined in the sense of convergence in mean for $f \in L^p[a,b]$ for all $p \geq \gamma$ [cf. Kwapien & Woyczynski (1992)]. Nayak, Pattanayak & Mishra (1987) considered random series (8) where, the random variables X_n are the Fourier coefficients of a stochastic process and a_n are Fourier coefficients of a function. They established convergence of the series to the stochastic integral $\int_a^b f(t)dX(t,\omega)$ in different stochastic sense, depending on the choice of the function f defined on $[a,b]$ whose Fourier coefficients are a_n.

These motivated us to explore RFS (7) in Hermite functions, with random coefficients X_n being Fourier–Hermite(FH) coefficients of a continuous stochastic process like the Wiener process and stable process. The random series is considered to be in orthogonal transformed Hermite functions and the random coefficients X_n are chosen to be independent, while dealing with the Wiener process $W(t,\omega)$. If the scalars $\tilde{a}_n$ represent the FH coefficients of an $L^2(\mathbb{R})$ function, the convergence of this series is proved to be in the quadratic mean. The convergence of a sequence of random variables in quadratic mean is defined as follows:

Definition 1 (cf. Lukacs (1975))**.** *A sequence of random variables $\{X_n\}$ is said to converge in the quadratic mean to a random variable X if all X_n and also X has finite-order moments and if* $\lim_{n\to\infty} E|X_n - X|^2 = 0$.

In more general the convergence in the r^{th} mean is defined as below:

Definition 2 (cf Lukacs (1975))**.** *A sequence of random variables $\{X_n\}$ is said to converge in the r^{th} mean to a random variable X if all X_n and also X have finite - moments of order $\gamma > 0$ and if* $\lim_{n\to\infty} E|X_n - X|^\gamma = 0$.

When the process is stable, the series is considered to be in orthogonal Hermite functions and the random variables X_n are chosen to be independent. The convergence of this series holds in the sense of mean if the scalars in the series are FH coefficients of an $L^2(\mathbb{R})$ function. The FT of the sum function of these series are computed, which exist in quadratic mean and mean, respectively.

The Hermite polynomials $H_n(t)$ of degree n are defined as $H_n(t) = (-1)^n e^{t^2}(\frac{d}{dt})^n\{e^{-t^2}\}$. These polynomials are orthogonal to one another in terms of the weighted inner product with the weight function e^{-t^2}, i.e.,

$$\int_{-\infty}^{\infty} H_m(t)H_n(t)e^{-t^2}dt = \sqrt{\pi}2^n n!\delta_{mn}, \tag{10}$$

where, δ_{mn} is the Kronecher's delta function. The normalized Hermite functions, also known as Hermite-Gaussian functions or simply Hermite functions of degree n ($n \in \mathbb{N}_0$) [cf. Celeghini & Del Olmo (2013, 2015); Celeghini (2019); Olver, Lozier, Boisvert & Clark (2010)], are defined as

$$\psi_n(t) := \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n(t)e^{-\frac{t^2}{2}}, \quad n \geq 0, \quad t \in \mathbb{R}, \tag{11}$$

such that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = \delta_{mn}. \tag{12}$$

The Hermite functions $\psi_n(t)$ provide a complete orthonormal framework on the Hilbert space $L^2(\mathbb{R})$[cf. Muckenhoupt (1970)].

Pollard (1948) showed that, if $f \in L^2(\mathbb{R})$ and $\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 e^{-t^2}dt\right)^{\frac{1}{2}}$, then

$$\| s_n - f \|_2 \to 0 \tag{13}$$

as $n \to \infty$ where the n^{th} partial sum of the series $\sum_{k=0}^{\infty} \tilde{a}_k \psi_k(t)$ is denoted by the symbol s_n. This conclusion was expanded to a wider class of functions by Askey & Wainger (1965). They discovered that for f in $L^p(\mathbb{R})$, $\frac{4}{3} < p < 4$,

$$\| f - \sum_{k=0}^{n} \tilde{a}_k \psi_k(t) \|_p \to 0, \tag{14}$$

as $n \to \infty$ where $\|f\|_p$ is the typical norm of L^p which is $\|f\|_p = \{\int_{-\infty}^{\infty} |f|^p dx\}^{\frac{1}{p}}$.

Pawlak & Stadtmüller (2008) extended this result further to $L^p(\mathbb{R})$, $p > 1$ and obtained that the series (1) converges in $L^p(\mathbb{R})$ for $t \in \mathbb{R}$ a.e..

This article is organized as follows: Section 2 deals with random series

$$\sum_{n=0}^{\infty} \tilde{b}_n \tilde{B}_n(\omega) \tilde{\psi}_n^{\alpha}(s), \tag{15}$$

associated with the Wiener process. The orthogonal functions considered to be the transformed Hermite functions $\tilde{\psi}_n^{\alpha}(s)$, which are orthogonal about a weight function $\mathbf{w}(s)$. The scalars $\tilde{b}_n$ represent the Fourier-transformed Hermite coefficients of a function f in the weighted space $L^2_{\mathbf{w}}(0,1)$, and the random variables $\tilde{B}_n(\omega)$ are the Fourier coefficients of the Wiener process concerning the transformed Hermite function, which are found to be independent.

The sum function is identified as the stochastic integral $\int_{-\infty}^{\infty} f(t) dX(t,\omega)$, which exists in the sense of quadratic mean. FT of this sum function is computed, which exists in quadratic mean. Computation of FT of the sum function of (15) requires finding FT of the transformed Hermite function. Since transformed Hermite functions are defined through generalized Hermite functions, the subsequent two subsections discuss the FT of generalized Hermite functions and transformed Hermite functions.

Section 3 is related to the symmetric stable process of index 2. The integral $\int_{-\infty}^{\infty} f(t) dX(t,\cdot)$ is shown to exist by using the fact that $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. If $f(t) = \psi_n(t)$ is the normalised Hermite function, then $\int_{-\infty}^{\infty} \psi_n(t) dX(t,\cdot)$ exists, which is denoted as $\tilde{A}_n$. It is shown that $\tilde{A}_n$ are independent, by using their characteristic functions(CF). The CF is computed in a wider sense by computing the CF of $\int_{-\infty}^{\infty} f(t) dX(t,\cdot)$ for f in $L^2(\mathbb{R})$. The convergence of a sequence of random variables in law required, which is defined below. The scalars $\tilde{a}_n$ are the FH coefficients of a function f in $L^2(\mathbb{R})$. FT of the sum function of the series is then determined, which exists in the sense of mean.

Definition 3 (cf. Lukacs (1975))**.** *Let $\{X_n\}$ be a sequence of random variables with distribution function $\{F_{X_n}\}$. $\{X_n\}$ is converges in law(or weakly) to X if distribution functions $\{F_{X_n}\}$ converges weakly to a distribution function $\{F_X\}$.*

2 Fourier transform of random Hermite series associated with Wiener process

This section deals with the series in transformed Hermite functions, $\tilde{\psi}_n^{\alpha}(t)$. If a function has a series expansion in transformed Hermite function of the form $\sum_{n=0}^{\infty} \tilde{b}_n \tilde{\psi}_n^{\alpha}(t)$, we can compute its FT as

$$\sum_{n=0}^{\infty} \tilde{b}_n \mathcal{F}(\tilde{\psi}_n^{\alpha}(t)), \tag{16}$$

where, $\mathcal{F}(\tilde{\psi}_n^{\alpha}(t))$ is the FT of $\tilde{\psi}_n^{\alpha}(t)$. The series (16) is said to be the FT of $f(t)$, denoted as $\mathcal{F}(f(t))$ if the infinite sum exists. To find FT of transformed Hermite function $\sum_{n=0}^{\infty} \tilde{b}_n \tilde{\psi}_n^{\alpha}(t)$, we have to go through generalised Hermite functions.

2.1 Fourier transform of generalized Hermite series:

Bao & Shen (2005); Guo, Shen & Xu (2003); Xiang & Wang (2013) describe the generalised Hermite function as

$$\psi_n^\alpha(t) = \frac{\sqrt{\alpha}}{\sqrt{2^n n!\sqrt{\pi}}} H_n(\alpha t) e^{-\frac{\alpha^2 t^2}{2}}, \qquad n \geq 0, t \in \mathbb{R} \tag{17}$$

which are normalized so that

$$\int_{-\infty}^{\infty} \psi_n^\alpha(t)\psi_m^\alpha(t)dt = \delta_{nm}. \tag{18}$$

The generalized Hermite functions $\psi_n^\alpha(t)$ are complete in $L^2(\mathbb{R})$. Hence for any function $g \in L^2(\mathbb{R})$, we have an associated expansion

$$g(t) := \sum_{n=0}^{\infty} b_n \psi_n^\alpha(t), \tag{19}$$

where,

$$b_n := \int_{-\infty}^{\infty} g(s)\psi_n^\alpha(s)ds, \qquad n \in \mathbb{N}_0, s \in \mathbb{R}. \tag{20}$$

For each given function $g \in L^2(\mathbb{R})$, the Fourier–Hermite transform(FHT) and fractional Fourier–Hermite transform(FrFHT) of (19) are

$$\mathcal{F}(g(t)) := \sum_{n=0}^{\infty} b_n \mathcal{F}(\psi_n^\alpha(t)). \tag{21}$$

and

$$\mathcal{F}^\beta(g(t)) := \sum_{n=0}^{\infty} b_n \mathcal{F}^\beta(\psi_n(t)) \tag{22}$$

respectively. The n^{th} eigenfunction of the Sturm–Liouville problem,

$$-f''(t) + \alpha^4 t^2 f(t) - \alpha^2(2k+1)f(t) = 0, \tag{23}$$

is the function $\psi_n^\alpha(t)$.
Since

$$\mathcal{F}(\psi_n(at+b)) := \frac{\lambda_n}{|a|} e^{-\frac{ibt}{a}} \psi_n(\frac{t}{a}),$$

$a \neq 0$, $b \in \mathbb{R}$, $t \in \mathbb{R}$ [cf. Bultheel & Martínez-Sulburah (2006); Liu & Liu (2007)] and $\psi_n^\alpha(t) := \sqrt{\alpha}\psi_n(\alpha t)$, it can be shown that

$$\mathcal{F}(\psi_n^\alpha(t)) := \frac{1}{\sqrt{\alpha}} \lambda_n \psi_n(\frac{t}{\alpha}), \qquad \alpha > 0. \tag{24}$$

The fractional FT of $\psi_n^\alpha(t)$ is

$$\mathcal{F}^\beta(\psi_n^\alpha(t)) := \sqrt{\alpha}\frac{\lambda_n^\beta}{|\alpha|} \psi_n(\frac{t}{\alpha}), \qquad \alpha > 0, 0 < \beta < 1. \tag{25}$$

Hence the FHT and FrFHT of $g \in L^2(\mathbb{R})$ becomes

$$\mathcal{F}(g(t)) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{\alpha}} b_n \lambda_n \psi_n(\frac{t}{\alpha}). \tag{26}$$

and

$$\mathcal{F}^\beta(g(t)) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{\alpha}} b_n \lambda_n^\beta \psi_n(\frac{t}{\alpha}). \tag{27}$$

2.2 Fourier transform of transformed Hermite series :

Generalised Hermite functions $\psi_n^\alpha(t)$ defined in the domain $\mathbb{R}$ are transformed to Hermite functions $\tilde{\psi}_n^\alpha(t)$ in the domain $(0,1)$ by composition of $\psi_n^\alpha(t)$ with a conformal map. Saadatmandi & Akbari (2017) considered the one-one conformal map as follows:

$$\tilde{\omega} = \varphi(z) \tag{28}$$

where $\varphi(z) = \ln(\frac{z}{1-z})$. The inverse of φ becomes

$$z = \frac{e^{\tilde{\omega}}}{1+e^{\tilde{\omega}}}.$$

The domain of φ is

$$D_E = \big\{z = t + is, \qquad t, s \in \mathbb{R} : |\arg(\frac{z}{1-z})| < d \leq \frac{\pi}{2}\big\}$$

and the range is

$$D_S = \big\{\tilde{\omega} = x + iy, \qquad x, y \in \mathbb{R} : |s| < d \leq \frac{\pi}{2}\big\},$$

for $d > 0$.

The real line interval between $(0,1)$ is mapped to the complete real line $\mathbb{R}$ by $\varphi(z)$. The range of φ^{-1} on the real line is

$$\mathcal{I} = (0,1).$$

The transformed Hermite functions $\tilde{\psi}_n^\alpha(x)$ are defined as

$$\tilde{\psi}_n^\alpha(x) \equiv \psi_n^\alpha \circ \phi(x) = \psi_n^\alpha(\phi(x)), \qquad \alpha > 0. \tag{29}$$

They defined the weighted space:

$$L^2_{\mathbf{w}}(\mathcal{I}) = \{g : \mathcal{I} \to \mathbb{R} \text{ is measurable and } \|g\|_{L^2_{\mathbf{w}}(\mathcal{I})} < \infty\},$$

with weighted function $\mathbf{w}(t) = \varphi^{'}(t) = \frac{1}{t(1-t)}$, which is a real-valued function that is integrable, non-negative, and defined over the interval $\mathcal{I}$ and

$$\|g\|_{L^2_{\mathbf{w}}(\mathcal{I})} := \big(\int_0^1 |g(t)|^2 \mathbf{w}(t) dt\big)^{\frac{1}{2}} = \big(\int_0^1 |g(t)\sqrt{\mathbf{w}(t)}|^2 dt\big)^{\frac{1}{2}} \tag{30}$$

is the norm induced by the inner product of the space $L^2_w(\mathcal{I})$. The system $\{\tilde{\psi}_n^\alpha(t)\}_{n=0}^\infty$ are mutually orthogonal over the interval $(0,1)$ with respect to this weight function $\mathbf{w}(t)$ [cf. Saadatmandi & Akbari (2017)]. Given that the system $\{\tilde{\psi}_n^\alpha(t)\}_{n=0}^\infty$ is complete in $L^2_{\mathbf{w}}(I)$, any function $\tilde{g} \in L^2_{\mathbf{w}}(0,1)$, can have the series expansion in transformed Hermite functions, i.e.,

$$\tilde{g}(t) := \sum_{n=0}^{\infty} \tilde{b}_n \tilde{\psi}_n^\alpha(t), \tag{31}$$

where,

$$\tilde{b}_n := \int_0^1 \tilde{g}(s)\tilde{\psi}_n^\alpha(s)\mathbf{w}(s) ds, \tag{32}$$

for $n \in \mathbb{N}_0$, $s \in \mathbb{R}$. This means that

$$\int_0^1 \big|\big(g(t) - \tilde{g}_n(t)\big)\big|^2 \mathbf{w}(t) dt = 0, \tag{33}$$

where, $\tilde{g}_n(t)$ is the partial sum of the series (31). The FHT and the FrFHT of the series (31) can be written as

$$\mathcal{F}(\tilde{g}(t)) := \sum_{n=0}^{\infty} \tilde{b}_n \mathcal{F}(\tilde{\psi}_n^{\alpha}(t)) \tag{34}$$

and

$$\mathcal{F}^{\beta}(\tilde{g}(t)) := \sum_{n=0}^{\infty} \tilde{b}_n \mathcal{F}^{\beta}(\tilde{\psi}_n^{\alpha}(t)). \tag{35}$$

The function $\tilde{\psi}_n^{\alpha}$, $n \in \mathbb{N}_0$ are the n^{th} eigen function of the unique singular Sturm–Liouville problem

$$-\{t(1-t)u^{'}(t)\}^{'} + \frac{\alpha^4\varphi^2(t)}{t(1-t)}u(t) - (2n+1)\frac{\alpha^2}{t(1-t)}u(t) = 0.$$

It is possible to calculate the FT of $\tilde{\psi}_n^{\alpha}(t)$ as $\frac{1}{\sqrt{\alpha}}\lambda_n\psi_n(\frac{\phi(t)}{\alpha})$. As a result, the FT of $\tilde{g}(t)$ is computed to be

$$\mathcal{F}(\tilde{g}(t)) = \sum_{n=0}^{\infty} \tilde{b}_n \frac{1}{\sqrt{\alpha}}\lambda_n\psi_n(\frac{\phi(t)}{\alpha}). \tag{36}$$

Similarly the FrFHT of transformed Hermite function can be computed as $\frac{1}{\sqrt{\alpha}}\lambda_n^{\beta}\psi_n(\frac{\phi(t)}{\alpha})$ and hence the FrFHT of $\tilde{g}(t)$ can be expressed as $\sum_{n=0}^{\infty} \tilde{b}_n \frac{1}{\sqrt{\alpha}}\lambda_n^{\beta}\psi_n(\frac{\varphi(t)}{\alpha})$.

2.3 Random Fourier transformed Hermite series associated with Wiener process:

The random series considered are in transformed Hermite functions. The random coefficients are related to the Wiener process. Let $W(s,\omega), s > 0$ be the Wiener process. Since

$$E\Big|\int_0^1 f(s)dW(s,\omega)\Big|^2 = c^2\int_0^1 |f(s)|^2 ds$$

for $f \in L^2(0,1)$, where c is a constant connected to the process's usual normal law of increment. The stochastic integral $\int_0^1 f(s)\sqrt{\mathbf{w}(s)}dW(s,\omega)$ is defined in quadratic mean for $f \in L^2_{\mathbf{w}}(0,1)$.

In particular if $f(s) = \tilde{\psi}_n^{\alpha}(s) \in L^2_{\mathbf{w}}(0,1)$ then the stochastic integrals

$$\tilde{B}_n(\omega) = \int_0^1 \tilde{\psi}_n^{\alpha}(s)\sqrt{\mathbf{w}(s)}dW(s,\omega), \qquad n \in \mathbb{N}_0 \tag{37}$$

exist and are random variables called the FH coefficients of $W(s,\omega)$. Since $\tilde{\psi}_n^{\alpha}(s)\sqrt{\mathbf{w}(s)}$ are continuous, the random variables $\tilde{B}_n(\omega)$ are normally distributed with mean 0 and finite variance[cf. Lukacs (1975), p. 148]. The normality of $\tilde{B}_n(\omega)$ implies their independentness by proving

$$E(\tilde{B}_n(\omega)\tilde{B}_m(\omega)) = 0$$

for $n \neq m$, in the following theorem.

Theorem 1. *The random variables $\tilde{B}_n(\omega)$, $n \in \mathbb{N}_0$(37) are independent.*

Proof. If a process $Y(s,\omega)$ has orthogonal increments and if $u, v \in L^2[a,b]$, then by Doob (1990)[see p. 427]

$$E\Big(\int_a^b u(s)dW(s,\omega)\overline{\int_a^b v(s)dW(s,\omega)}\Big) = \int_a^b u(s)\overline{v(s)}ds,$$

where $\overline{v(s)}$ is the complex conjugate of $v(s)$. By applying this fact, we get

$$\begin{aligned} E(\tilde{B}_n(\omega)\tilde{B}_m(\omega)) &= E\big(\int_0^1 \tilde{\psi}_n^\alpha(s)\sqrt{\mathbf{w}(s)}dW(s,\omega)\int_0^1 \tilde{\psi}_m^\alpha(s)\sqrt{\mathbf{w}(s)}dW(s,\omega)\big) \\ &= \int_0^1 \tilde{\psi}_n^\alpha(s)\tilde{\psi}_m^\alpha(s)\mathbf{w}(s)ds, \end{aligned}$$

since the Wiener process $W(s,\omega)$ has orthogonal increments. As

$$\int_0^1 \tilde{\psi}_n^\alpha(s)\tilde{\psi}_m^\alpha(s)\mathbf{w}(s)ds = \delta_{nm},$$

where, $\delta_{nm} = 0$ for $n \neq m$ [cf. Saadatmandi & Akbari (2017)], this proves that $\tilde{B}_n(\omega)$ are independent random variables. □

The convergence of the random series

$$\sum \tilde{b_n}\tilde{B}_n(\omega)\tilde{\psi}_n^\alpha(t) \tag{38}$$

is shown below. Here the FH coefficients of $\tilde{g}$ are defined as

$$\tilde{b}_n = \int_0^1 \tilde{\psi}_n^\alpha(s)\tilde{g}(s)\mathbf{w}(s)ds, \qquad s > 0.$$

Suppose the n^{th} partial sum of the series $\sum_{k=0}^{\infty}\tilde{b}_k\tilde{B}_k(\omega)\tilde{\psi}_k^\alpha(t)$ is

$$\tilde{S}_n(t,\omega) := \sum_{k=0}^{n}\tilde{b}_k\tilde{B}_k(\omega)\tilde{\psi}_k^\alpha(t).$$

Substituting the integral form of $\tilde{B}_n(\omega)$ in $\tilde{S}_n(t,\omega)$, we get

$$\begin{aligned} \tilde{S}_n(t,\omega) &:= \sum_{k=0}^{n}\tilde{b_k}\big(\int_0^1 \tilde{\psi}_k^\alpha(s)\sqrt{\mathbf{w}(s)}dW(s,\omega)\big)\tilde{\psi}_k^\alpha(t) \\ &= \int_0^1 \sum_{k=0}^{n}\tilde{b_k}\tilde{\psi}_k^\alpha(s)\tilde{\psi}_k^\alpha(t)\sqrt{\mathbf{w}(s)}dW(s,\omega). \end{aligned}$$

Denote the partial sum of the transformed Hermite series $\sum_{n=0}^{\infty}\tilde{b}_n\tilde{\psi}_n^\alpha(t)$ of $\tilde{g} \in L^2_{\mathbf{w}}(0,1)$ as

$$\tilde{t}_n(t) := \sum_{k=0}^{n}\tilde{b}_k\tilde{\psi}_k^\alpha(t).$$

Since $\tilde{\psi}_n^\alpha(t)$ are bounded in $(0,1)$, $\sum_{k=0}^{\infty}\tilde{b}_k\tilde{\psi}_k^\alpha(s)\tilde{\psi}_k^\alpha(t)$ exists and let it be denoted as $\tilde{g}(s,t)$. Let us denote the partial sum

$$\sum_{k=0}^{n}\tilde{b}_k\tilde{\psi}_k^\alpha(s)\tilde{\psi}_k^\alpha(t)$$

as $\tilde{t}_n(s,t)$.

Theorem 2. *Let $\tilde{g} \in L^2_{\mathbf{w}}(0,1)$ and $W(t,\omega)$, $t > 0$, $\omega \in \Omega$ be the Wiener process. Then the random series (38) in transformed Hermite functions converges in quadratic mean to the integral $\int_0^1 \tilde{g}(s,t)dW(s,\omega)$.*

Proof. The partial sum of the series (38) is

$$\tilde{S}_n(t,\omega) = \int_0^1 \tilde{t}_n(s,t)\sqrt{\mathbf{w}(s)}dW(s,\omega).$$

We are aware that $\int_0^1 \tilde{g}(s,t)\sqrt{\mathbf{w}(s)}dW(s,\omega)$ exists in the notion of convergence in quadratic mean. Now,

$$\begin{aligned} & E\left(\left|\int_0^1 \tilde{g}(s,t)\sqrt{\mathbf{w}(s)}dW(s,\omega) - \tilde{S}_n(t,\omega)\right|^2\right) \\ = \ & E\left(\left|\int_0^1 \tilde{g}(s,t)\sqrt{\mathbf{w}(s)}dW(s,\omega) - \int_0^1 \tilde{t}_n(s,t)\sqrt{\mathbf{w}(s)}dW(s,\omega)\right|^2\right) \\ = \ & E\left(\left|\int_0^1 (\tilde{g}(s,t) - \tilde{t}_n(s,t))\sqrt{\mathbf{w}(s)}dW(s,\omega)\right|^2\right) \\ = \ & \int_0^1 \left|(\tilde{g}(s,t) - \tilde{t}_n(s,t))\right|^2 \mathbf{w}(s)ds, \end{aligned}$$

which converges to 0 by (33). Therefore, the theorem is proved. □

Let us use the notation $\tilde{F}(t,\omega)$ to represent the sum function $\int_0^1 \tilde{g}(s,t)\sqrt{\mathbf{w}(s)}dW(s,\omega)$ of the series $\sum_{n=0}^{\infty} \tilde{b}_n \tilde{B}_n(\omega)\tilde{\psi}_n^{\alpha}(t)$.
Now it is possible to calculate the Fourier transform of $\tilde{F}(t,\omega)$ as

$$\mathcal{F}(\tilde{F}(t,\omega)) := \sum_{n=0}^{\infty} \tilde{b}_n \tilde{B}_n(\omega)\mathcal{F}(\tilde{\psi}_n^{\alpha}(t)) = \sum_{n=0}^{\infty} \tilde{b}_n \tilde{B}_n(\omega)\frac{1}{\sqrt{\alpha}}\lambda_k \tilde{\psi}_n(\frac{\phi(t)}{\alpha}), \tag{39}$$

where λ_k are eigenvalues of the transformed Hermite function with absolute value one. The following theorem establishes the convergence of the series (39).

Theorem 3. *If $\tilde{g} \in L^2_{\mathrm{w}}(0,1)$ and $\tilde{b}_n$, $\tilde{B}_n$ are defined as in (39) and (37) then the series*

$$\sum_{n=0}^{\infty} \tilde{b}_n \tilde{B}_n(\omega)\frac{1}{\sqrt{\alpha}}\lambda_n \psi_n(\frac{\phi(t)}{\alpha}), \tag{40}$$

converges in the sense of quadratic mean to the stochastic integral $\int_0^1 \mathcal{F}(\tilde{g}(s,t))dW(s,\omega)$.

Proof. Denote the n^{th} partial sum of the series (40) as

$$\tilde{T}_n(t,\omega) := \sum_{k=0}^{n} \tilde{b}_k \tilde{B}_k(\omega)\frac{1}{\sqrt{\alpha}}\lambda_k \psi_k(\frac{\phi(t)}{\alpha}). \tag{41}$$

Its integral form is $\tilde{T}_n(t,\omega) = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{1}{\sqrt{\alpha}}\tilde{b_k}\tilde{\psi}_k^{\alpha}(s)\lambda_k\psi_k(\frac{\phi(t)}{\alpha})\sqrt{\mathbf{w}(s)}\right)dW(s,\omega)$.
Denote $\sum_{k=0}^{\infty} \tilde{b}_k \tilde{\psi}_k^{\alpha}(s)\tilde{\psi}_k^{\alpha}(t)$ as $\tilde{g}(s,t)$ as in Theorem 1.

The FT of $\tilde{g}(s,t)$ can be computed to be $\sum_{k=0}^{\infty} \tilde{b}_k \tilde{\psi}_k^{\alpha}(s)\mathcal{F}(\tilde{\psi}_k^{\alpha}(t)) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{\alpha}}\tilde{b_k}\tilde{\psi}_k^{\alpha}(s)\lambda_k\psi_k(\frac{\phi(t)}{\alpha})$. Let the partial sum of $\mathcal{F}(\tilde{g}(s,t))$ be $\sum_{k=0}^{n} \frac{1}{\sqrt{\alpha}}\tilde{b_k}\tilde{\psi}_k^{\alpha}(s)\lambda_k\psi_k(\frac{\phi(t)}{\alpha})$. Denote it as $\mathcal{F}(\tilde{t}_n(s,t)$, where $\tilde{t}_n(s,t) := \sum_{k=0}^{n} \tilde{b}_k \tilde{\psi}_k^{\alpha}(s)\tilde{\psi}_k^{\alpha}(t)$ is the partial sum of $\tilde{g}(s,t)$. Now

$$\tilde{T}_n(t,\omega) = \int_0^1 \mathcal{F}(\tilde{t}_n(s,t))\sqrt{\mathbf{w}(s)}dW(s,\omega).$$

Since FT of a function in $L^2_{\mathbf{w}}(0,1)$ is in $L^2_{\mathbf{w}}(0,1)$, the stochastic integral $\int_0^1 \mathcal{F}(\tilde{t}_n(s,t))\sqrt{\mathbf{w}(s)}dW(s,\omega)$ exists in the sense of convergence in quadratic mean. Now,

$$\begin{aligned}
& E\left|\int_0^1 \mathcal{F}(\tilde{g}(s,t))\sqrt{\mathbf{w}(s)}dW(s,\omega) - \tilde{T}_n(t,\omega)\right|^2 \\
= \;& E\left|\int_0^1 \mathcal{F}(\tilde{g}(s,t))\sqrt{\mathbf{w}(s)}dW(s,\omega) - \int_0^1 \mathcal{F}(\tilde{t}_n(s,t))\sqrt{\mathbf{w}(s)}dW(s,\omega)\right|^2 \\
= \;& \int_0^1 \left|\mathcal{F}(\tilde{g}(s,t)) - \mathcal{F}(\tilde{t}_n(s,t))\right|^2 \mathbf{w}(s)ds \\
= \;& \int_0^1 \left|\mathcal{F}(\tilde{g}(s,t) - \tilde{t}_n(s,t))\right|^2 \mathbf{w}(s)ds \\
= \;& \int_0^1 \left|\tilde{g}(s,t) - \tilde{t}_n(s,t)\right|^2 \mathbf{w}(s)ds
\end{aligned}$$

which converges to 0. Hence, the theorem is proved. □

Since the series (22) converges we say $\sum_{n=0}^{\infty} \tilde{b}_n \tilde{B}_n(\omega)\frac{1}{\sqrt{\alpha}}\lambda_n\psi_n(\frac{\phi(t)}{\alpha})$ is the FHT of the random function $\tilde{F}(t,\omega)$. Define (40) to be the random FHT of the function $\tilde{g} \in L^2_{\mathbf{w}}(0,1)$, associated with Wiener process, which exists in quadratic mean.

3 Fourier transform of random Hermite series associated with symmetric stable process

Let $f \in L^2(\mathbb{R})$ and $X(t,.)$ be a stable process of index 2. The existence of the stochastic integral $\int_{-\infty}^{\infty} f(t)dX(t,\cdot)$ is established in Theorem 4 below. The following lemma is used to prove this.

Lemma 1. *[cf. Pattanayak & Sahoo (2005)] If $X(t,\cdot)$ for $t \in \mathbb{R}$ is a symmetric stable process with independent increment of index γ, for $1 < \gamma \leq 2$ and $g \in L^p[a,b]$, $p \geq \gamma$, then*

$$E\left(\left|\int_a^b g(t)dX(t,\cdot)\right|\right) \leq \frac{4}{\pi(\gamma-1)}\int_a^b |g(t)|^\gamma dt + \frac{2}{\pi}\int_{|u|>1} \frac{1-\exp(-c|u|^\gamma \int_a^b |g(t)|^\gamma dt)}{u^2}du.$$

Theorem 4. *If $X(t,\cdot)$ for $t \in \mathbb{R}$ is a symmetric stable process of index 2 and $f \in L^2(\mathbb{R})$, then the stochastic integral $\int_{-\infty}^{\infty} f(t)dX(t,\cdot)$ exists in the sense of convergence in mean.*

Proof. We are aware that $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. So for $f \in L^2(\mathbb{R})$ there exist a sequence of functions $\{g_k\}$ in $C_c(\mathbb{R})$ such that $(f - g_k) \in L^2(\mathbb{R})$ and $\|f - g_k\|_2$ approaches 0 as $k \to \infty$. Consider two functions g_m and g_n from this sequence $\{g_k\}$. Without loss of generality assume that the compact support of g_m and g_n to be $[a,b]$ and [c, d] respectively. Let $[p,q]$ be the smallest closed sub - interval of $\mathbb{R}$ which contains both $[a,b]$ and $[c,d]$. Now both g_m and g_n can be considered to be in $L^2[p,q]$. Since g_m and g_n are continuous, the stochastic integrals

$$\int_p^q g_k(t)dX(t,\cdot), \;\; k \; = \; m, \; n,$$

exist in the sense of mean[see Pattanayak & Sahoo (2005)]. Consequently the integrals $\int_{-\infty}^{\infty} g_k(t)dX(t,\cdot)$ exist, as the support of g_k lies in $[p,q]$. Denote $Y_i := \int_p^q g_i(t)dX(t,\cdot)$, $i = m,n$. Application of Lemma

1 for $\gamma = 2$ gives

$$\begin{aligned} E\big|Y_n - Y_m\big| &= E\left(\left|\int_p^q g_n(t)dX(t,\cdot) - \int_p^q g_m(t)dX(t,\cdot)\right|\right) \\ &= E\Big(\Big|\int_p^q (g_n(t) - g_m(t))dX(t,\cdot)\Big|\Big) \\ &\le \frac{4}{\pi}\int_p^q |g_n(t) - g_m(t)|^2 dt + \frac{2}{\pi}\int_{|u|>1} \frac{1 - \exp\big(-c|u|^2 \int_p^q |g_n(t) - g_m(t)|^2 dt\big)}{u^2} du \\ &\le \frac{4}{\pi}\int_{-\infty}^{\infty} |g_n(t) - g_m(t)|^2 dt + \frac{2}{\pi}\int_{|u|>1} \frac{1 - \exp\big(-c|u|^2 \int_{-\infty}^{\infty} |g_n(t) - g_m(t)|^2 dt\big)}{u^2} du. \end{aligned}$$

Since $\|g_n - g_m\|_2 = \int_{-\infty}^{\infty} |g_n - g_m|^2 dt$ approaches 0 as $m, n \to \infty$ and the integrand in the 2^{nd} integral is dominated by the integrable function $\frac{1}{u^2}$ over $(-\infty, -1]$ and $[1, \infty)$, the $2nd$ integral converges to 0 by dominated convergence theorem. Thus we obtain

$$\lim_{m,n\to\infty} E|Y_n - Y_m| = 0.$$

So Y_n is a Cauchy sequence in the sense of mean, and hence, there exists a random variable Y such that $E|Y_n - Y| = 0$. This Y is independent of the choice of the sequence of functions g_n. In fact, if another sequence h_n in $C_c(\mathbb{R})$ converges to f i.e.,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |h_n(t) - f(t)|^2 dt = 0 \text{ as } n \to \infty.$$

Then $\lim_{n\to\infty} \int_{-\infty}^{\infty} |h_n(t) - g_n(t)|^2 dt$ converges to 0. Thus by Lemma 1 we obtain

$$\lim_{n\to\infty} E\left(\left|\int_{-\infty}^{\infty} h_n(t)dX(t,\cdot) - Y\right|\right) = 0.$$

Hence the stochastic integral $\int_{-\infty}^{\infty} g_n(t)dX(t,\cdot)$ converges uniquely to Y, in the sense of mean. Define this random variable Y to be the stochastic integral, $Y = \int_{-\infty}^{\infty} f(t)dX(t,\cdot)$. □

If $f(t)$ is the Hermite function $\psi_n(t)$, then the integral $\int_{-\infty}^{\infty} \psi_n(t)dX(t,\cdot)$ will exist, and is a random variable. Give the notation

$$\tilde{A}_n = \int_{-\infty}^{\infty} \psi_n(t)dX(t,\cdot). \tag{42}$$

These $\tilde{A}_n$ are independent. It is established in Theorem 7, by demonstrating that the CF of $(\tilde{A}_n + \tilde{A}_m)$ is equal to the product of a CF of $\tilde{A}_n$ and the CF of $\tilde{A}_m$. In more general for $f \in L^2(\mathbb{R})$, the CF of $\int_{-\infty}^{\infty} f(t)dX(t,\cdot)$ is computed in the following theorem.

Theorem 5. *The CF of* $\int_{-\infty}^{\infty} f(t)dX(t,\cdot)$ *is*

$$\exp\big(-c|u|^2 \int_{-\infty}^{\infty} |f(t)|^2 dt\big)$$

where $f(t) \in L^2(\mathbb{R})$.

In the proof we will use the continuity theorem which is stated below.

Theorem 6 (Continuity Theorem (cf. Lukacs (1975), p. 15))**.** *The sequence* $\{F_{X_n}\}$ *of distribution functions converges weakly to a distribution function* F_X *if and only if, the corresponding sequence of CF* $\{f_n(t)\}$ *converges for every* t *to a function* $f(t)$ *which is continuous at* $t = 0$*. This function* $f(t)$ *is then the CF of the limiting distribution* F_X*.*

Proof of Theorem 5. As we know $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exist a sequence of functions $\{g_k\}$ in $C_c(\mathbb{R})$ for $f \in L^2(\mathbb{R})$ such that $\|g_k - f\|_2 \to 0$. Further it is known that the stochastic integrals $\int_{-\infty}^{\infty} g_k dX(t,\cdot)$ and $\int_{-\infty}^{\infty} f(t)dX(t,\omega)$ exist in mean by Theorem 4. Denote these random variables as $Y_k := \int_{-\infty}^{\infty} g_k(t)dX(t,\cdot)$ and $Y := \int_{-\infty}^{\infty} f(t)dX(t,\cdot)$. Since g_n has compact support the CF of $Y_n := \exp\big(-c|u|^2 \int_{-\infty}^{\infty} |g_n|^2 dt\big)$[cf. Pattanayak & Sahoo (2005)]. As $f \in L^2(\mathbb{R})$, it is true that $\|f\|_2 \leq R$, $\|g\|_2 \leq R$ for some constant R and

$$\int_{-\infty}^{\infty} \big||g_n(t)|^2 - |f(t)|^2\big| dt \leq 4R \int_{-\infty}^{\infty} |g_n(t) - f(t)|^2,$$

which implies

$$\int_{-\infty}^{\infty} |g_n(t)|^2 dt \text{ approaches } \int_{-\infty}^{\infty} |f(t)|^2 dt \text{ as } n \to \infty.$$

This further implies

$$\exp\big(-c|u|^2 \int_{-\infty}^{\infty} |g_k(t)|^2 dt\big) \text{ approaches } \exp\big(-c|u|^2 \int_{-\infty}^{\infty} |f(t)|^2 dt\big), \text{ as } k \to \infty.$$

LHS is the CF of Y_n, which converges to the continuous function on the RHS. By Continuity Theorem(6), RHS is the CF of the limiting function of Y_k, which is Y. This proves that the CF of $\int_{-\infty}^{\infty} f(t)dX(t,\cdot)$ is

$$\exp\big(-c|u|^2 \int_{-\infty}^{\infty} |f(t)|^2 dt\big).$$

□

Theorem 7. *The random variables $\tilde{A}_n = \int_{-\infty}^{\infty} \psi_n(t)dX(t,\cdot)$ associated with $X(t,\cdot)$ are independent if $X(t,\cdot)$, $t \in \mathbb{R}$ is a symmetric stable process of index 2.*

Proof. By Theorem 5, the CF of $\tilde{A}_n$ is $\exp\big(-c|u|^2 \int_{-\infty}^{\infty} |\psi_n(t)|^2 dt\big)$. Hence, the CF of $(\tilde{A}_n + \tilde{A}_m)$ is $\exp\big(-c|u|^2 \int_{-\infty}^{\infty} |\psi_n(t) + \psi_m(t|^2 dt\big)$, whereas the product of CF of $\tilde{A}_n$ and the CF of $\tilde{A}_m$ is $\exp\big(-c|u|^2 \int_{-\infty}^{\infty} |\psi_n(t)|^2 dt\big) \exp\big(-c|u|^2 \int_{-\infty}^{\infty} |\psi_m(t)|^2 dt\big) = \exp\big(-c|u|^2 \int_{-\infty}^{\infty} (|\psi_n(t)|^2 + |\psi_m(t)|^2) dt\big)$ which are equal. Hence, $\tilde{A}_n$ are independent random variables. □

Consider the random Hermite series

$$\sum_{n=0}^{\infty} \tilde{a}_n \tilde{A}_n \psi_n(t), \tag{43}$$

where ψ_n are normalized Hermite–Gaussian functions, the scalars $\tilde{a}_n$ are the FH coefficients of a function f in $L^2(\mathbb{R})$ defined as $\int_{-\infty}^{\infty} f(t)\psi_n(t)dt$ and $\tilde{A}_n$ are random variables defined as in (42).

The inequality in the following lemma is needed to prove the convergence of the RFH series (43).

Lemma 2. *Let f be any function in $L^2(\mathbb{R})$ and $X(t,\cdot)$, $t \in \mathbb{R}$ be a symmetric stable process of index 2, then*

$$E\left(\left|\int_{-\infty}^{\infty} f(t)dX(t,\cdot)\right|\right) \leq \frac{4}{\pi} \int_{-\infty}^{\infty} |f(t)|^2 dt + \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp\big(-c|u|^2 \int_{-\infty}^{\infty} |f(t)|^2 dt\big)}{u^2} du.$$

To prove it, we require the following two results.

Lemma 3 (cf. Shiryayev, (1984), p. 341). *A stable random variable $X(t,\cdot)$ always satisfies the inequality $E|X|^r < \infty$ for all $r \in (0,\gamma)$, $0 < \gamma \leq 2$.*

Lemma 4 (cf. Chow & Teicher (2005), p. 285)**.** *If ψ is the CF of a random variable X, then $E|X| = \int |X| dF_X = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-Re\psi(u)}{u^2} du$, where F_X is the distribution function of X.*

Proof of Lemma 2. We know that $\int_{-\infty}^{\infty} f(t)dX(t,\cdot)$ exists in the sense of mean (Theorem 4). Now using Lemma 3 and 4, we have

$$\begin{aligned} E\left(\left|\int_{-\infty}^{\infty} f(t)dX(t,\cdot)\right|\right) &= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1-Re\psi(u)}{u^2}du \\ &= \frac{2}{\pi}\int_{|u|\le 1}\frac{1-Re\psi(u)}{u^2}du + \frac{2}{\pi}\int_{|u|>1}\frac{1-Re\psi(u)}{u^2}du. \end{aligned}$$

Here,

$$\begin{aligned} \int_{|u|\le 1}\frac{1-Re\psi(u)}{u^2}du &= \int_{-1}^{1}\frac{1-\exp\left(-c|u|^2\int_{-\infty}^{\infty}|f(t)|^2dt\right)}{u^2}du \\ &\le \int_{-1}^{1}\frac{|u|^2\int_{-\infty}^{\infty}|f(t)|^2dt}{u^2}du \quad (\because\ 1 - e^{-x} < x \text{ for } x > 0) \\ &= 2\int_{0}^{1}|u|^{2-2}du\int_{-\infty}^{\infty}|f(t)|^2dt \\ &= 2\int_{-\infty}^{\infty}|f(t)|^2dt \end{aligned}$$

Hence,

$$E\left(\left|\int_{-\infty}^{\infty} f(t)dX(t,\cdot)\right|\right) \le \frac{4}{\pi}\int_{-\infty}^{\infty}|f(t)|^2dt + \frac{2}{\pi}\int_{|u|>1}\frac{1-\exp\left(-c|u|^2\int_{-\infty}^{\infty}|f(t)|^2dt\right)}{u^2}du.$$

□

The following theorem establishes the convergence of the series (43) in the sense of mean, if $\tilde{a}_n$ are the FH coefficients of $f \in L^2(\mathbb{R})$. Let,

$$S_n(t,\cdot) := \sum_{k=0}^{n}\tilde{a}_k\tilde{A}_k\psi_k(t)$$

be the n^{th} partial sum of the Fourier–Hermite series (43). Now

$$\begin{aligned} S_n(t,\cdot) &= \sum_{k=0}^{n}\tilde{a}_k\tilde{A}_k\psi_k(t) \\ &= \sum_{k=0}^{n}\tilde{a}_k\left(\int_{-\infty}^{\infty}\psi_k(s)dX(s,\cdot)\right)\psi_k(t) \\ &= \int_{-\infty}^{\infty}\Big(\sum_{k=0}^{n}\tilde{a}_k\psi_k(s)\psi_k(t)\Big)dX(s,.) \end{aligned}$$

Since $\psi_n(t)$ are bounded by $\pi^{-\frac{1}{4}}$[cf. Szego (1939); Indritz (1961)], $\sum_{k=0}^{\infty}\tilde{a}_k\psi_k(s)\psi_k(t)$ exists. Denote it as $f(s,t)$. Let $s_n(s,t) := \sum_{k=0}^{n}\tilde{a}_k\psi_k(s)\psi_k(t)$ be its partial sum.

Theorem 8. *If $X(s,\cdot), s \in \mathbb{R}$ is a symmetric stable process of index 2, then the series (43) with random coefficients $\tilde{A}_n$ defined as in (42), converges in the mean to the stochastic integral $\int_{-\infty}^{\infty} f(s,t)dX(s,\cdot)$ if $\tilde{a}_n$ are the FH coefficients of a function $f \in L^2(\mathbb{R})$.*

Proof. For every $f \in L^2(\mathbb{R})$, the FH series expansion of f is $\sum_{n=-\infty}^{\infty} \tilde{a}_n \psi_n(t)$ [cf. Askey & Wainger (1965)]. Let its partial sum be $s_n(t) := \sum_{k=0}^{n} \tilde{a}_k \psi_k(t)$. Now the integral form of $S_n(t, \cdot)$ is

$$S_n(t, \cdot) = \int_{-\infty}^{\infty} s_n(s,t) dX(s, \cdot).$$

According to Theorem 4, $\int_{-\infty}^{\infty} f(s,t) dX(s, \cdot)$ exists in the sense of convergence in mean. Now by applying Lemma 2,

$$\begin{aligned} & E\left(\left|\int_{-\infty}^{\infty} f(s,t)dX(s,\cdot) - S_n(t,\cdot)\right|\right) \\ = \; & E\left(\left|\int_{-\infty}^{\infty} f(s,t)dX(s,\cdot) - \int_{-\infty}^{\infty} s_n(s,t)dX(s,\cdot)\right|\right) \\ = \; & E\left(\left|\int_{-\infty}^{\infty} (f(s,t) - s_n(s,t)dX(t,\cdot)\right)\right. \\ \leq \; & \frac{4}{\pi}\int_{-\infty}^{\infty} |(f(s,t) - s_n(s,t)|^2 ds \\ + \; & \frac{2}{\pi}\int_{|u|>1} \frac{1 - \exp\big(-c|u|^2 \int_{-\infty}^{\infty} |(f(s,t) - s_n(s,t)|^2 ds\big)}{u^2} du. \end{aligned}$$

Since the integral $\int_{-\infty}^{\infty} |f(s,t) - s_n(s,t)|^2 ds$ tends to 0 as $n \to \infty$ and the integrand in the second integral is dominated by the integrable function $\frac{1}{u^2}$, the theorem is proved by applying (14) for $p = 2$. □

Let the sum function $\int_{-\infty}^{\infty} f(s,t) dX(s, \cdot)$ of the series (43) be denoted as $F(t, \cdot)$. The series (43) is the random Fourier–Hermite series(RFHS), associated with the symmetric stable process, $X(t, \cdot)$ of index 2.

Theorem 9. *Let $X(s, \cdot)$ be the symmetric stable process of index 2 and $\tilde{A}_k$ be its FH coefficients. For $f \in L^2(\mathbb{R})$, if $\tilde{a}_k$ are its FH coefficients, and $\mathcal{F}$ is the FT operator, then the series*

$$\sum_{k=0}^{\infty} \tilde{a}_k \tilde{A}_k \lambda_k \psi_k(t), \tag{44}$$

converges in the mean to the stochastic integral

$$\int_{-\infty}^{\infty} \mathcal{F}(f(s,t)) dX(s, \cdot), \tag{45}$$

where $\lambda_k = e^{-\frac{ik\pi}{2}}$, $k \in \mathbb{N}_0$ are the eigen values of the FT $\mathcal{F}(\psi_n(t))$.

Proof. Let

$$\hat{S}_n(t, \cdot) := \sum_{k=0}^{n} \tilde{a}_k \tilde{A}_k \lambda_k \psi_k(t)$$

be the n^{th} partial sum of the series (44). Denote the partial sum of the series $\sum_{k=0}^{\infty} \tilde{a}_k \lambda_k \psi_k(t)$ as $\mathcal{F}(s_n(t)) := \sum_{k=0}^{n} \tilde{a}_k \lambda_k \psi_k(t)$ which is the FT of $s_n(t)$ and let

$$\mathcal{F}(s_n(s,t)) := \sum_{k=0}^{n} \tilde{a}_k \lambda_k \psi_k(s) \psi_k(t).$$

In integral form,

$$\begin{aligned}\hat{S}_n(t,\cdot) &= \sum_{k=0}^{n} \tilde{a}_k \lambda_k \tilde{A}_k \psi_k(t) \\ &= \sum_{k=0}^{n} \tilde{a}_k \lambda_k \left(\int_{-\infty}^{\infty} \psi_k(s) dX(s,\cdot) \right) \psi_k(t) \\ &= \int_{-\infty}^{\infty} \mathcal{F}(s_n(s,t)) dX(s,\cdot).\end{aligned}$$

Since the integral $\int_{-\infty}^{\infty} \left|\mathcal{F}(f(s,t)) - \mathcal{F}(s_n(s,t))\right|^2 ds$ converges to 0 for $f \in L^2(\mathbb{R})$, it can be shown that $E\left(\left| \int_{-\infty}^{\infty} \mathcal{F}[f(s,t)] dX(s,\cdot) - \hat{S}_n(t,\cdot) \right| \right)$ converges to 0 as $n \to \infty$ using the same argument as in the previous theorem. Hence, it is proved that the random series (44) converges in mean to $\int_{-\infty}^{\infty} \mathcal{F}[f(s,t)] dX(s,\cdot)$. □

Define the series (44) as the RFHT of the function $f \in L^2(\mathbb{R})$ associated with the symmetric stable process, $X(t,\cdot)$, of index 2.

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4 Conflicts of Interest

The authors declare no conflict of interest.

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