ISSN 0971-0388

Aligarh Journal of Statistics
Vol. 44(2024), 127-142

Lindley's Approximation in Bayesian Parameter Estimation for the Quadratic Transmuted Exponential Model

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[Received on November, 2023. Accepted on June, 2024]

ABSTRACT

In the contemporary era, a notable emphasis exists on the practice of generalizing probability distributions, a common approach observed across diverse research domains. This practice entails extending pre-existing baseline probability models to encapsulate and adeptly analyze the intricacies inherent in data. The quadratic transmuted family of distributions, distinguished by its amalgamation of the cumulative distribution function and the quantile function of the base-line distribution. The current study is dedicated to scrutinizing the behaviors exhibited by parameters in the exponential distribution in relation to the quadratic rank transmuted map. Bayesian methodology is the chosen avenue for estimating these parameters, with a deliberate selection of non-informative priors considering symmetric and asymmetric loss functions, facilitating the estimation of the rate and transmuted parameters within the Quadratic Transmuted Exponential Distribution. Since Bayes estimator in closed-form is unfeasible, the study leverages Lindley's approximation as a computational tool for determining the Bayes estimators through a comprehensive Monte Carlo simulation study, particularly emphasizing evaluating their posterior risks. The study applies these research findings in a practical context by addressing a real-life data application, thereby underscoring the tangible significance and applicability of the research outcomes.

1. Introduction

Exponential distribution is a valuable tool for analyzing positively skewed lifetime data and has extensive applications in fields such as reliability engineering and life testing. It is particularly well-suited for situations characterized by a constant hazard rate. However, it may not be suitable for modeling real-life phenomena that exhibit bathtub failure rates. It's worth noting that the Exponential distribution represents a special case within the broader frameworks of both the Weibull and Gamma distributions as noted by Gupta and Kundu (1999), Gupta and Kundu (2007), Oguntunde and Adejumo (2015). When assuming the rate parameter θ, the Cumulative Distribution Function (CDF) of the exponential random variable X is expressed as follows:

$$G(x) = \begin{cases} 1 - e^{-\theta x}, & x > 0 \\ 0, & otherwise \end{cases} \tag{1.1}$$

The corresponding Probability Density Function (PDF) of X is,

$$g(x;\theta) = \theta e^{-\theta x}; x > 0;\ \theta > 0 \tag{1.2}$$

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The concept of transmutation involves the functional composition of the cumulative distribution function with the inverse cumulative distribution function (quantile function) of a baseline distribution. This concept was originally introduced by Shaw and Buckley (2007), who pioneered the quadratic transmuted family of distributions, known for its flexibility in modeling skewed distributions. Shaw and Buckley (2009) also introduced the quadratic transmutation map method as a means to generate novel families of distributions by introducing a parameter to the baseline distributions (Jabarali *et al.* (2024)).

This approach has led to the creation of various flexible probability distributions. For instance, Merovci (2013a) introduced the transmuted exponentiated exponential distribution, while Elbatal *et al.* (2013a), Elbatal *et al.* (2013) introduced the transmuted generalized inverted exponential distribution and transmuted generalized linear exponential distribution, respectively. Tian *et al.* (2014) introduced the transmuted linear exponential distribution, analyzing its properties and comparing estimators through simulation. Owoloko *et al.* (2015) derived the transmuted exponential distribution and employed the method of least squares for parameter estimation, along with discussing structural properties like mean, moments, and quantile functions. Khan *et al.* (2017a) proposed the transmuted generalized exponential distribution and explored its mathematical properties. Rahman *et al.* (2018a), Rahman *et al.* (2018c) introduced the cubic transmuted exponential distribution, investigating its mathematical characteristics.

Furthermore, Okorie and Akpanta (2019) introduced the transmuted generalized inverted exponential distribution and derived its properties. Most recently, Ghost *et al* (2021) introduced the transmuted generalized linear exponential distribution, elucidating its mathematical properties, comparing estimators using simulation data, and suggesting real-life applications. It is noteworthy that transmuted distributions have proven to be highly flexible and applicable in diverse fields such as survival analysis, reliability, economics, engineering, insurance, and bio-statistics.

In this study, the authors undertake a Bayesian analysis of the quadratic transmuted exponential distribution, a formulation initially developed by Owoloko *et al.* (2015). Notably, there is a gap in the existing literature concerning investigating parameter behaviors within the QTED. Thus, the primary objective of this research is to comprehensively examine the performance of these parameters through Bayesian estimation techniques, employing Lindley's approximation. Within this Bayesian framework, non-informative uniform priors are adopted to derive Bayes estimates of the parameters. These estimates are subsequently scrutinized under various loss functions, different sets of parameters, and diverse sample sizes, with a particular focus on mean square errors for comparative assessment.

1.1 Quadratic Transmuted Exponential Distribution

The PDF of a quadratic transmuted random variable X is given as,

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]; x > 0; |\lambda| \leq 1 \qquad (1.3)$$

where, λ is the transmuted parameter.

Also, the CDF of X is given as;

$$F(x) = \begin{cases} (1+\lambda)G(x) - \lambda[G(x)]^2, & x > 0 \\ 0, & otherwise \end{cases} \qquad (1.4)$$

G(x) and g(x) is the CDF and PDF of the baseline distribution respectively. According to Shaw and Buckley (2009), if λ = 0, f(x) and F(x) reduces to the baseline distributions.

To obtain the PDF of QTED, we substitute equation (1.2) in equation (1.3), which gives

$$f(x) = \theta e^{-\theta x}\left[1 + \lambda - 2\lambda(1 - e^{-\theta x})\right]$$

After some simplifications, we get,

$$f(x) = \theta e^{-\theta x}\left[1 - \lambda + 2\lambda e^{-\theta x}\right];\ x > 0; |\lambda| \leq 1 \tag{1.5}$$

where, θ is the rate parameter and λ is the transmuted parameter.

To obtain the CDF of QTED, we substitute equation (1.1) in equation (1.4).

$$F(x) = (1 + \lambda)(1 - e^{-\theta x}) - \lambda\left[1 - e^{-\theta x}\right]^2$$

After some algebraic simplifications, we get,

$$F(x) = \begin{cases}(1 - e^{-\theta x})(1 + \lambda)e^{-\theta x}, & x > 0\\ 0, & otherwise\end{cases} \tag{1.6}$$

In particular, if λ = 0, the transmuted exponential distribution reduces to the ordinary exponential distribution.

The reliability function and hazard rate function of the mentioned distribution are as follows:

$$R(x) = \lambda e^{-2\theta x} + (1 - \lambda)e^{-\theta x};\ x > 0; |\lambda| \leq 1$$

and

$$h(x) = \frac{\theta\left[1 - \lambda + 2\lambda e^{-\theta x}\right]}{\left[1 - \lambda + \lambda e^{-\theta x}\right]}$$

The Mean Time to Failure (MTTF) of QTED is,

$$MTTF = \frac{2 - \lambda}{2\theta}$$

The paper is structured into six sections, with this introduction as the first. Section 2 elaborates on the methodology of Bayesian approaches, providing a foundational understanding of the analytical techniques employed. Section 3 details Bayes estimates of QTED with different loss functions. Section 4 is dedicated to a comprehensive simulation study conducted under different prior distributions. Section 5 introduces real-world data used for the assessment of Bayes estimators. Finally, Section 6 offers concluding remarks summarizing the findings and insights derived from this research.

2. Methodology

This section delves into the Bayesian parameter estimation method, a statistical approach that integrates sample data with prior knowledge about the parameters before observing the sample. In Bayesian methodology, the model parameters are treated as random variables, and an appropriate probability distribution is selected to represent these parameters based on the available prior information. The choice of prior distribution is a crucial step in the Bayesian framework. In situations where there is limited or vague prior knowledge about the parameters, it is common practice to employ non-informative priors. For this research, the authors opt to use a Uniform prior for the parameters denoted as $\Theta=(\theta_1, \theta_2,...,\theta_k)$. The joint posterior distribution is derived by combining the likelihood of the sample data with the chosen prior distribution. This joint posterior distribution is represented as:

$$g(\Theta|x) = \frac{L(x;\Theta)\pi(\Theta)}{\int_0^\infty L(x;\Theta)\pi(\Theta)d\Theta} \quad (2.1)$$

Where: $\pi(\Theta)$ represents the joint prior distribution of $\Theta=(\theta_1, \theta_2,...,\theta_k)$. $L(x;\Theta)$ is the likelihood function. $g(\Theta|x)$ denotes the joint posterior distribution, which combines the information from the sample data and the prior (Jabarali & Kannan *et al.* (2015); Kannan *et al.* (2016); Jabarali *et al.* (2016))

2.1 Bayes Estimators of the Parameters under Different Loss Functions

In the Bayesian approach for estimating unknown parameters, the specification of a loss function becomes necessary. A loss function, denoted as L(α,$\hat{\alpha}$), can assume either symmetric or asymmetric characteristics (Jabarali *et al.* (2018)). The choice of an appropriate loss function is not governed by specific rules. If a loss function provides an equal footing for both overestimation and underestimation, it is categorized as symmetric in nature. A well-known example of a symmetric loss function is the Squared Error Loss Function (SELF), which was introduced by Legendre (1805). An alternative to symmetric loss functions is the use of asymmetric precautionary loss functions (PLF), as introduced by Norstrom (1996). These loss functions are designed to mitigate underestimation and yield conservative estimators. Another example of an asymmetric loss function is the linear exponential (LINEX) Loss function, originally proposed by Varian (1975). According to Soliman (2006), overestimation occurs when the parameter 'm' is greater than zero, while underestimation occurs when 'm' is less than zero. The LINEX loss function closely approximates the SELF when 'm' is approximately equal to zero. Table 1 presents the expressions for Bayes estimators under the aforementioned loss functions for an unknown parameter α.

Table 1: Bayes estimators under different loss functions.

Loss Function	**Bayes Estimator**	**Posterior Risk**
SELF=$(\hat{\alpha}-\alpha)^2$	$\hat{\alpha}_{BS} = E(\alpha\|x)$	$E(\alpha^2\|x) - [E(\alpha\|x)]^2$
PLF =$\frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}$	$\hat{\alpha}_{BP} = \sqrt{E(\alpha^2\|x)}$	$2\left[\sqrt{E(\alpha^2\|x)} - E(\alpha\|x)\right]$
LINEX=$e^{m(\hat{\alpha}-\alpha)} - m(\hat{\alpha}-\alpha) - 1$	$\hat{\alpha}_{BL} = \frac{-1}{m} log[E(e^{-m\alpha}\|x)]$	$log[E(e^{-m\alpha}\|x)] + mE(\alpha\|x)$

2.2 Lindley's Approximation Method

The calculation of Bayes estimators under both symmetric and asymmetric loss functions entails working with integrals in a ratio form, as described in equation (2.1). These integrals do not lend themselves to analytical solutions. To estimate the parameters effectively, the researchers in this study employ Lindley's approximation procedure, as introduced by Lindley (1980). The posterior expectation of an arbitrary function $\hat{u}(\theta,\lambda)$ is evaluated as

$$I(x) = E[u(\theta,\lambda)|x] = \frac{\int u(\theta,\lambda)exp[L(\theta,\lambda)+\rho(\theta,\lambda)]d(\theta,\lambda)}{\int exp[L(\theta,\lambda)+\rho(\theta,\lambda)]d(\theta,\lambda)} \quad (2.2)$$

where u(θ,λ) is a function of θ and λ only, L(θ,λ) is the log-likelihood function and ρ(θ,λ) is the log of joint prior of θ and λ.

According to Lindley (1980) (2.2) is approximated aymptotically by;

$$I(x) = u(\hat{\theta},\hat{\lambda}) + 0.5[(\hat{u}_{\theta\theta} + 2\hat{u}_{\theta}\hat{\rho}_{\theta})\hat{\sigma}_{\theta\theta} + (\hat{u}_{\lambda\theta} + 2\hat{u}_{\lambda}\hat{\rho}_{\theta})\hat{\sigma}_{\lambda\theta} + (\hat{u}_{\theta\lambda} + 2\hat{u}_{\theta}\hat{\rho}_{\lambda})\hat{\sigma}_{\theta\lambda} + (\hat{u}_{\lambda\lambda} + 2\hat{u}_{\lambda}\hat{\rho}_{\lambda})\hat{\sigma}_{\lambda\lambda}] + 0.5\big[(\hat{u}_{\theta}\hat{\sigma}_{\theta\theta} + \hat{u}_{\lambda}\hat{\sigma}_{\theta\lambda})\big(\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{L}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda} + \hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta} + \hat{L}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}\big) + (\hat{u}_{\theta}\hat{\sigma}_{\lambda\theta} + \hat{u}_{\lambda}\hat{\sigma}_{\lambda\lambda})(\hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda} + \hat{L}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta} + \hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda})\big] \quad (2.3)$$

$\hat{\theta}$ and $\hat{\lambda}$ are the MLE's of θ and λ respectively.

Where,

$$\hat{u}_{\theta} = \frac{\partial u(\hat{\theta},\hat{\lambda})}{\partial\hat{\theta}} \quad \hat{u}_{\lambda} = \frac{\partial u(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}} \quad \hat{u}_{\theta\theta} = \frac{\partial^2 u(\hat{\theta},\hat{\lambda})}{\partial\hat{\theta}^2} \quad \hat{u}_{\lambda\lambda} = \frac{\partial^2 u(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}^2}$$

$$\hat{u}_{\theta\lambda} = \frac{\partial^2 u(\hat{\theta},\hat{\lambda})}{\partial\hat{\theta}\partial\hat{\lambda}} \quad \hat{u}_{\lambda\theta} = \frac{\partial^2 u(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\theta}} \quad \hat{L}_{\theta\theta\theta} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\theta}^3} \quad \hat{L}_{\lambda\lambda\lambda} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}^3}$$

$$\hat{L}_{\theta\lambda\theta} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\theta}\partial\hat{\lambda}\partial\hat{\theta}} \quad \hat{L}_{\lambda\theta\theta} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\theta}\partial\hat{\theta}} \quad \hat{L}_{\lambda\lambda\theta} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\lambda}\partial\hat{\theta}} \quad \hat{L}_{\theta\lambda\lambda} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\theta}\partial\hat{\lambda}\partial\hat{\theta}}$$

$$\hat{L}_{\lambda\theta\lambda} = \frac{\partial^3 L(\hat{\theta},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\theta}\partial\hat{\lambda}}$$

3. Bayes Estimators of QTED

Maximum likelihood estimators are computed to estimate the parameters of the QTED within the framework of Bayesian estimation. The procedure for deriving these maximum likelihood estimators is elaborated upon below:

3.1 Maximum Likelihood Estimator of QTED

Let X_1, X_2,…X_n be a random sample of size n from the transmuted exponential distribution, with θ and λ as the parameters. The likelihood function of θ and λ is obtained as,

$$L(x_1, x_2, \dots, x_n|\theta,\lambda) = \theta^n e^{-\sum x_i\theta} \prod_{i=1}^{n}\left[1 - \lambda + 2\lambda e^{-x_i\theta}\right] \quad (3.1)$$

The corresponding log-likelihood function is given by;

$$l = logL = nlog(\theta) - \sum_{i=1}^{n} x_i\theta + \sum_{i=1}^{n} log\left[1 - \lambda + 2\lambda e^{-x_i\theta}\right] \quad (3.2)$$

Differentiate (3.2) with respect to θ and λ

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i - 2\lambda\theta \sum \frac{e^{-x_i\theta}}{[1 - \lambda + 2\lambda e^{-x_i\theta}]}$$

$$\frac{\partial l}{\partial\lambda} = \sum_{i=1}^{n} \frac{2e^{-x_i\theta} - 1}{[1 - \lambda + 2\lambda e^{-x_i\theta}]}$$

Equating the equations to zero and solving the resulting non-linear system of equations provides the MLEs for the parameters of QTED.

Given that this system of equations lacks an analytical solution, a numerical approach is required. The researchers employ the Newton-Raphson method to obtain the desired MLEs from the given equations. All computational procedures are carried out using the R statistical software.

3.2 Posterior Distribution using Uniform Prior

In this section, we have employed a Bayesian approach to estimate the unknown parameters of the QTED. The prior distributions for θ and λ are specified as follows:

$\theta \propto 1, \forall\, \theta \in (0,\infty)$ and $\lambda \propto 1, \forall\, \lambda \in [-1, 1]$

Given the assumption of independence between the prior distributions of the parameters, we can express the joint prior distribution of the parameters θ and λ as follows:

$$\pi(\theta,\lambda) \propto 1;\ \theta > 0\ and\ \lambda \in [-1,1] \tag{3.3}$$

Therefore, by utilizing (2.1), (3.1), and (3.3), the joint posterior distribution of the parameters θ and λ, given the data x under a uniform prior, can be expressed as:

$$g(\theta,\lambda|x) = \frac{L(x;\theta,\lambda)\pi(\theta,\lambda)}{\int_0^\infty \int_{-1}^1 L(x;\theta,\lambda)\pi(\theta,\lambda)d\lambda d\theta}$$

$$= \frac{\theta^n e^{-\sum x_i\theta} \prod_{i=1}^n [1-\lambda+2\lambda e^{-x_i\theta}]}{\int_0^\infty \int_{-1}^1 \theta^n e^{-\sum x_i\theta} \prod_{i=1}^n [1-\lambda+2\lambda e^{-x_i\theta}]\ d\lambda d\theta} \tag{3.4}$$

3.3 Bayes Estimators under Squared Error Loss Function (SELF)

3.3.1 Bayes Estimators of θ under SELF

The Bayes estimator of θ under SELF presented in Table [1]. The Bayes Estimator $\hat{\theta}_{BS}$, a function u=u(θ,λ) of the unknown parameters of TED under SELF is the posterior mean. From (2.2),

$$\hat{\theta}_{BS} = E[\theta|x]$$

$$= \frac{\int u(\theta,\lambda)\, exp[L(\theta,\lambda)+\rho(\theta,\lambda)]d(\theta,\lambda)}{\int exp[L(\theta,\lambda)+\rho(\theta,\lambda)]d(\theta,\lambda)} \tag{3.5}$$

where, $u(\theta, \lambda) = \theta$

$$L(\theta,\lambda) = nlog(\theta) - \theta\sum_{i=1}^n x_i + \sum_{i=1}^n \log[1-\lambda+2\lambda e^{-x_i\theta}]$$

$\rho(\theta, \lambda) = \log(1) = 0$

It can be easily verified that;

$$\hat{u}_\theta = 1;\ \hat{u}_{\theta\theta} = 0;\ \hat{u}_\lambda = \hat{u}_{\lambda\lambda} = \hat{u}_{\lambda\theta} = \hat{u}_{\theta\lambda} = 0$$

$$\hat{L}_\theta = \frac{n}{\theta} - \sum_{i=1}^n x_i - \sum_{i=1}^n \frac{2\lambda x_i e^{-\theta x_i}}{1-\lambda+2\lambda e^{-x_i\theta}}$$

$$\hat{L}_{\theta\theta} = \frac{-n}{\theta^2} + \sum_{i=1}^n \frac{2\lambda x_i^2 e^{-\theta x_i}}{1-\lambda+2\lambda e^{-x_i\theta}} - \sum_{i=1}^n \frac{4\lambda^2 x_i^2 e^{-2\theta x_i}}{[1-\lambda+2\lambda e^{-x_i\theta}]^2}$$

$$\hat{L}_{\theta\theta\theta} = \frac{2n}{\theta^3} - \sum_{i=1}^{n} \frac{2\lambda {x_i}^3 e^{-\theta x_i}}{1-\lambda+2\lambda e^{-x_i\theta}} + \sum_{i=1}^{n} \frac{12\lambda^2 {x_i}^3 e^{-2\theta x_i}}{[1-\lambda+2\lambda e^{-x_i\theta}]^2} - \sum_{i=1}^{n} \frac{16\lambda^3 {x_i}^3 e^{-3\theta x_i}}{[1-\lambda+2\lambda e^{-x_i\theta}]^3}$$

$$\hat{L}_{\lambda} = \sum_{i=1}^{n} \frac{2e^{-\theta x_i}-1}{1-\lambda+2\lambda e^{-x_i\theta}}$$

$$\hat{L}_{\lambda\lambda} = -\sum_{i=1}^{n} \frac{(2e^{-\theta x_i}-1)^2}{(1-\lambda+2\lambda e^{-x_i\theta})^2}$$

$$\hat{L}_{\theta\lambda\lambda} = \sum_{i=1}^{n} \frac{4x_i e^{-\theta x_i}(2e^{-\theta x_i}-1)}{(1-\lambda+2\lambda e^{-x_i\theta})^3}$$

$$\hat{L}_{\theta\theta\lambda} = \sum_{i=1}^{n} \frac{2{x_i}^2 e^{-\theta x_i}}{(1-\lambda+2\lambda e^{-x_i\theta})^2} - \sum_{i=1}^{n} \frac{8\lambda {x_i}^2 e^{-2\theta x_i}}{(1-\lambda+2\lambda e^{-x_i\theta})^3}$$

$\rho_\theta = \rho_\lambda = 0$; $\hat{\sigma}_{\theta\theta} = \frac{-1}{\hat{L}_{\theta\theta}}$; $\hat{\sigma}_{\lambda\lambda} = \frac{-1}{\hat{L}_{\lambda\lambda}}$; *Since*, θ and λ *are independent*, $\hat{\sigma}_{\theta\lambda} = 0$.

Using (2.3) and the above u-terms, L-terms and ρ terms, the Bayes Estimator of θ under SELF is,

$$\hat{\theta}_{BS} = \hat{\theta} + 0.5\left[\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}^2\right] \tag{3.6}$$

3.3.2 Bayes Estimators of λ under SELF

The Bayes estimator of λ under SELF presented in Table 1. Then

$$\hat{\lambda}_{BS} = E[\lambda|x]$$

$$= \frac{\int u(\theta,\lambda)\, exp[L(\theta,\lambda)+\rho(\theta,\lambda)]d(\theta,\lambda)}{\int \exp[L(\theta,\lambda)+\rho(\theta,\lambda)]d(\theta,\lambda)} \tag{3.7}$$

where, u(θ, λ) = λ, $L(\theta,\lambda)$ and $\rho(\theta,\lambda)$ are the same in (3.5).

It can be easily verified that; $\hat{u}_\lambda = 1$; $\hat{u}_{\lambda\lambda} = 0$; $\hat{u}_\theta = \hat{u}_{\theta\theta} = \hat{u}_{\lambda\theta} = \hat{u}_{\theta\lambda} = 0$

Following the procedure discussed in 3.3.1, we get,

$$\hat{\lambda}_{BS} = \hat{\lambda} + 0.5\left[\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}^2\right] \tag{3.9}$$

3.4 Bayes Estimators under PLF

3.4.1 Bayes Estimators of θ under PLF

Applying the same Lindley's approach here as in (2.2) with u(θ, λ) = θ^2;
$\hat{u}_\theta = 2\theta$; $\hat{u}_{\theta\theta} = 2$; $\hat{u}_\lambda = \hat{u}_{\lambda\lambda} = \hat{u}_{\lambda\theta} = \hat{u}_{\theta\lambda} = 0$.
$L(\theta,\lambda)$ and $\rho(\theta,\lambda)$ are the same in (3.5).
Following the procedure discussed in 3.3.1, we get,

$$E[\theta^2|x] = \hat{\theta}^2 + \hat{\sigma}_{\theta\theta} + \hat{\theta}\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{\theta}\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}^2$$

The Bayes estimator of θ under PLF is

$$\hat{\theta}_{BP} = \sqrt{\hat{\theta}^2 + \hat{\sigma}_{\theta\theta} + \hat{\theta}\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{\theta}\hat{L}_{\theta\theta\theta}\hat{\sigma}^2_{\theta\theta}} \quad (3.9)$$

3.4.2 Bayes Estimators of λ under PLF

Applying the same Lindley's approach here as in (2.2) with $u(\theta, \lambda) = \lambda^2$;
$\hat{u}_\lambda = 2\lambda$; $\hat{u}_{\lambda\lambda} = 2$; $\hat{u}_\lambda = \hat{u}_{\lambda\lambda} = \hat{u}_{\lambda\theta} = \hat{u}_{\theta\lambda} = 0$.
$L(\theta, \lambda)$ and $\rho(\theta, \lambda)$ are the same in (3.5).

Following the procedure discussed in 3.3.1, we get,

$$E[\lambda^2|x] = \hat{\lambda}^2 + \hat{\sigma}_{\lambda\lambda} + \hat{\theta}\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{\theta}\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}^2_{\lambda\lambda}$$

The Bayes estimator of θ under PLF is

$$\hat{\lambda}_{BP} = \sqrt{\hat{\lambda}^2 + \hat{\sigma}_{\lambda\lambda} + \hat{\theta}\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{\theta}\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}^2_{\lambda\lambda}} \quad (3.10)$$

3.5 Bayes Estimators under LINEX

3.5.1 Bayes Estimators of θ under LINEX

Applying the same Lindley's approach here as in (2.2) with $u(\theta, \lambda) = e^{-m\theta}$;
$\hat{u}_\theta = -me^{-m\theta}$; $\hat{u}_{\theta\theta} = m^2e^{-m\theta}$; $\hat{u}_\lambda = \hat{u}_{\lambda\lambda} = \hat{u}_{\lambda\theta} = \hat{u}_{\theta\lambda} = 0$.
$L(\theta, \lambda)$ and $\rho(\theta, \lambda)$ are the same in (3.5).

Following the procedure discussed in 3.3.1, we get,

$$E\left[e^{-m\theta}\middle|x\right] = e^{-m\hat{\theta}} + 0.5(m^2e^{-m\hat{\theta}}\hat{\sigma}_{\theta\theta} - 0.5me^{-m\hat{\theta}}(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\theta\theta\theta}\hat{\sigma}^2_{\theta\theta})$$

The Bayes estimator of θ under LINEX is

$$\hat{\theta}_{BL} = \frac{-1}{m}log\left[e^{-m\hat{\theta}} + 0.5(m^2e^{-m\hat{\theta}}\hat{\sigma}_{\theta\theta} - 0.5me^{-m\hat{\theta}}(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + +\hat{L}_{\theta\theta\theta}\hat{\sigma}^2_{\theta\theta})\right] \quad (3.11)$$

3.5.2 Bayes Estimators of λ under LINEX

Applying the same Lindley's approach here as in (2.2) with $u(\theta, \lambda) = e^{-m\lambda}$;
$\hat{u}_\lambda = -me^{-m\lambda}$; $\hat{u}_{\lambda\lambda} = m^2e^{-m\lambda}$; $\hat{u}_\theta = \hat{u}_{\theta\theta} = \hat{u}_{\lambda\theta} = \hat{u}_{\theta\lambda} = 0$.
$L(\theta, \lambda)$ and $\rho(\theta, \lambda)$ are the same in (3.5).
Following the procedure discussed in 3.3.1, we get,

$$E\left[e^{-m\lambda}\middle|x\right] = e^{-m\hat{\lambda}} + 0.5(m^2e^{-m\hat{\lambda}}\hat{\sigma}_{\lambda\lambda} - 0.5me^{-m\hat{\lambda}}(\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\lambda\lambda}\hat{\sigma}^2_{\lambda\lambda})$$

The Bayes estimator of λ under LINEX is

$$\hat{\lambda}_{BL} = \frac{-1}{m}log\left[e^{-m\hat{\lambda}} + 0.5(m^2e^{-m\hat{\lambda}}\hat{\sigma}_{\lambda\lambda} - 0.5me^{-m\hat{\lambda}}(\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\lambda\lambda}\hat{\sigma}^2_{\lambda\lambda})\right] \quad (3.12)$$

4. Simulation Study

In this section, the estimated values of the rate parameter θ and the transmuted parameter λ for both maximum likelihood estimation and Bayesian estimation employing uniform prior information under the SELF, PLF and Linear-LINEX loss function has been presented. The simulation study is conducted to assess the performance of the Bayesian estimators derived in Section 3. The authors have selected sample sizes of n=10, 50, 100, and 150, along with a different range of parameter values, including θ=0.5 and 1.5, λ=-0.9, -0.5, 0.5, and 1, and m=-1 and +1. These simulations have been iterated 1000 times using the R software. The generated samples are utilized to compute Bayes estimates and posterior risks. The simulation study results are thoroughly detailed in Tables 2-6, providing insights into the performance of the estimators under various scenarios and parameter configurations.

Table 2: Maximum Likelihood Estimate values and MSE of θ and λ.

		n = 10		n = 50		n = 100		n = 150	
θ	λ	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$
0.5	**-0.9**	0.5089 (0.0212)	-0.8119 (0.1369)	0.5024 (0.0040)	-0.8780 (0.0283)	0.5008 (0.0018)	-0.8934 (0.0127)	0.5016 (0.0012)	-0.8961 (0.0014)
	-0.5	0.5422 (0.0395)	-0.5265 (0.2573)	0.5042 (0.0097)	-0.5008 (0.1411)	0.5042 (0.0041)	-0.5089 (0.0577)	0.5031 (0.0029)	-0.5042 (0.0409)
	0.5	0.5449 (0.0576)	0.5449 (0.2204)	0.5118 (0.0210)	0.5036 (0.1247)	0.5031 (0.0161)	0.5190 (0.1081)	0.5006 (0.0129)	0.5214 (0.0919)
	1	0.5991 (0.0657)	0.9036 (0.0496)	0.5629 (0.0208)	0.8981 (0.0491)	0.5526 (0.0142)	0.9147 (0.0403)	0.5434 (0.0121)	0.9201 (0.0392)
1.5	**-0.9**	1.5446 (0.1943)	-0.8147 (0.1228)	1.500 (0.0342)	-0.8742 (0.0305)	1.5071 (0.0170)	-0.8916 (0.0136)	1.5028 (0.0126)	-0.8945 (0.0098)
	-0.5	1.6418 (0.3669)	-0.5524 (0.2557)	1.5224 (0.0802)	-0.4947 (0.1179)	1.5073 (0.0369)	-0.5019 (0.0612)	1.5100 (0.0252)	-0.5039 (0.0387)
	0.5	1.3384 (0.3194)	0.7912 (0.1687)	1.5081 (0.1486)	0.5258 (0.1371)	1.5027 (0.1304)	0.5186 (0.1101)	1.5008 (0.1076)	0.5125 (0.0979)
	1	1.8032 (0.6113)	0.9019 (0.0509)	1.6733 (0.1939)	0.9059 (0.0467)	1.6472 (0.1266)	0.9149 (0.0413)	1.6231 (0.1003)	0.9237 (0.0355)

Table 3: Bayes estimate and its Posterior Risks of θ and λ under SELF.

		n = 10		n = 50		n = 100		n = 150	
θ	λ	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$
0.5	**-0.9**	0.5506 (0.0150)	-0.6578 (0.3352)	0.5119 (0.0029)	-0.8013 (0.0402)	0.5055 (0.0014)	-0.8470 (0.0155)	0.5048 (0.0010)	-0.8633 (0.0102)
	-0.5	0.5872 (0.0195)	-0.3665 (0.2637)	0.5129 (0.0038)	-0.4512 (0.0510)	0.5086 (0.0019)	-0.4831 (0.0246)	0.5059 (0.0014)	-0.4871 (0.0167)
	0.5	0.5763 (0.0371)	0.4560 (0.2583)	0.5182 (0.0064)	0.4754 (0.0513)	0.5062 (0.0030)	0.5034 (0.0245)	0.5026 (0.0020)	0.5108 (0.0165)
	1	0.5889 (0.0406)	0.7441 (0.2475)	0.5581 (0.0070)	0.8232 (0.0392)	0.5495 (0.0034)	0.8690 (0.0164)	0.5412 (0.0021)	0.8870 (0.0109)
1.5	**-0.9**	1.6718 (0.1254)	-0.6667 (0.3528)	1.5280 (0.0270)	-0.7975 (0.0397)	1.5214 (0.0131)	-0.8456 (0.0170)	1.5123 (0.0085)	-0.8615 (0.0090)
	-0.5	1.7798 (0.1765)	-0.3907 (0.2549)	1.5485 (0.0345)	-0.4460 (0.0518)	1.5204 (0.0168)	-0.4764 (0.0254)	1.5187 (0.0113)	-0.4868 (0.0169)
	0.5	1.3701 (0.2187)	0.6499 (0.2777)	1.5262 (0.0553)	0.4959 (0.0495)	1.5120 (0.0270)	0.5028 (0.0249)	1.5071 (0.0178)	0.5022 (0.0168)
	1	1.7727 (0.3544)	0.7419 (0.2757)	1.6573 (0.0630)	0.8286 (0.0407)	1.6379 (0.0296)	0.8688 (0.0182)	1.6162 (0.0194)	0.8904 (0.0096)

Table 4: Bayes estimate and its Posterior Risks of θ and λ under PLF.

		n = 10		n = 50		n = 100		n = 150	
θ	λ	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$
0.5	**-0.9**	0.5641 (0.0270)	0.8763 (3.0682)	0.5147 (0.0056)	0.8260 (3.2546)	0.5069 (0.0028)	0.8561 (3.4062)	0.5058 (0.0020)	0.8692 (3.4650)
	-0.5	0.6036 (0.0328)	0.6309 (1.9948)	0.5166 (0.0074)	0.5046 (1.9116)	0.5105 (0.0038)	0.5079 (1.9820)	0.5073 (0.0028)	0.5040 (1.9822)
	0.5	0.6076 (0.0626)	0.6828 (0.4536)	0.5243 (0.0122)	0.5266 (0.1024)	0.5092 (0.0060)	0.5272 (0.0476)	0.5046 (0.0040)	0.5267 (0.0318)
	1	0.6224 (0.0670)	0.8951 (0.3020)	0.5643 (0.0124)	0.8467 (0.0470)	0.5526 (0.0062)	0.8784 (0.0188)	0.5431 (0.0038)	0.8931 (0.0122)
1.5	**-0.9**	1.7089 (0.0742)	0.8929 (3.1192)	1.5368 (0.0176)	0.8220 (3.2390)	1.5257 (0.0086)	0.8556 (3.4024)	1.5151 (0.0056)	0.8667 (3.4564)
	-0.5	1.8287 (0.0978)	0.6384 (2.0582)	1.5596 (0.0222)	0.5007 (1.8934)	1.5259 (0.0110)	0.5024 (1.9576)	1.5224 (0.0074)	0.5039 (1.9814)
	0.5	1.4477 (0.1552)	0.8367 (0.3736)	1.5442 (0.0360)	0.5435 (0.0952)	1.5209 (0.0178)	0.5270 (0.0484)	1.5130 (0.0118)	0.5187 (0.0330)
	1	1.8700 (0.1946)	0.9089 (0.3340)	1.6762 (0.0378)	0.8528 (0.0484)	1.6469 (0.0180)	0.8792 (0.0208)	1.6222 (0.0120)	0.8958 (0.0108)

Table 5: Bayes estimate and its Posterior Risks (in brackets) of θ and λ under LINEX for m=-1.

		n = 10		n = 50		n = 100		n = 150	
θ	λ	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$
0.5	**-0.9**	0.5571 (0.0065)	-0.5089 (0.1489)	0.5134 (0.0015)	-0.7823 (0.0190)	0.5063 (0.0008)	-0.8402 (0.0068)	0.5053 (0.0005)	-0.8582 (0.0051)
	-0.5	0.5969 (0.0097)	-0.2484 (0.1181)	0.5148 (0.0019)	-0.4274 (0.0238)	0.5095 (0.0009)	-0.4709 (0.0122)	0.5066 (0.0007)	-0.4787 (0.0084)
	0.5	0.5936 (0.0178)	0.5913 (0.1257)	0.5214 (0.0032)	0.5006 (0.0252)	0.5077 (0.0015)	0.5158 (0.0124)	0.5037 (0.0011)	0.5189 (0.0081)
	1	0.6088 (0.0199)	0.8898 (0.1457)	0.5615 (0.0034)	0.8447 (0.0215)	0.5511 (0.0016)	0.8775 (0.0085)	0.5422 (0.0010)	0.8926 (0.0056)
1.5	**-0.9**	1.7257 (0.0539)	-0.5099 (0.1568)	1.5408 (0.0128)	-0.7768 (0.0207)	1.5279 (0.0065)	-0.8380 (0.0076)	1.5166 (0.0043)	-0.8571 (0.0044)
	-0.5	1.8541 (0.0743)	-0.2718 (0.1189)	1.5652 (0.0167)	-0.4222 (0.0238)	1.5287 (0.0083)	-0.4640 (0.0124)	1.5244 (0.0057)	-0.4786 (0.0082)
	0.5	1.4711 (0.1010)	0.7871 (0.1372)	1.5530 (0.0268)	0.5218 (0.0259)	1.5253 (0.0133)	0.5155 (0.0127)	1.5160 (0.0089)	0.5106 (0.0084)
	1	1.9377 (0.1650)	0.8797 (0.1378)	1.6879 (0.0306)	0.8509 (0.0223)	1.6528 (0.0149)	0.8771 (0.0083)	1.6258 (0.0096)	0.8956 (0.0052)

Table 6: Bayes estimate and its Posterior Risks (in brackets) of θ and λ under LINEX for m=+1.

		n = 10		n = 50		n = 100		n = 150	
θ	λ	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$	$\hat{\theta}_{ML}$	$\hat{\lambda}_{ML}$
0.5	**-0.9**	0.5439 (0.0067)	-0.8326 (0.1748)	0.5105 (0.0014)	-0.8221 (0.0208)	0.5048 (0.0007)	-0.8556 (0.0086)	0.5043 (0.0005)	-0.8683 (0.0050)
	-0.5	0.5769 (0.0103)	-0.5111 (0.1446)	0.5109 (0.0020)	-0.4780 (0.0268)	0.5076 (0.0010)	-0.4959 (0.0128)	0.5054 (0.0005)	-0.4956 (0.0085)
	0.5	0.5580 (0.0183)	0.3392 (0.1168)	0.5151 (0.0031)	0.4508 (0.0246)	0.5047 (0.0015)	0.4913 (0.0121)	0.5016 (0.0010)	0.5027 (0.0081)
	1	0.5706 (0.0183)	0.5998 (0.1443)	0.5544 (0.0037)	0.8042 (0.0190)	0.5478 (0.0017)	0.8614 (0.0076)	0.5401 (0.0011)	0.8816 (0.0054)
1.5	**-0.9**	1.5979 (0.0739)	-0.8596 (0.1929)	1.5148 (0.0132)	-0.8179 (0.0204)	1.5146 (0.0068)	-0.8537 (0.0081)	1.5078 (0.0045)	-0.8668 (0.0053)
	-0.5	1.6818 (0.0980)	-0.5329 (0.1422)	1.5308 (0.0177)	-0.4731 (0.0271)	1.5118 (0.0086)	-0.4894 (0.0130)	1.5129 (0.0058)	-0.4954 (0.0086)
	0.5	1.2584 (0.1117)	0.5387 (0.1112)	1.4989 (0.0273)	0.4716 (0.0243)	1.4983 (0.0137)	0.4907 (0.0121)	1.4980 (0.0091)	0.4939 (0.0083)
	1	1.6124 (0.1603)	0.6023 (0.1396)	1.6274 (0.0299)	0.8125 (0.0161)	1.6234 (0.0145)	0.8609 (0.0079)	1.6068 (0.0094)	0.8854 (0.0050)

The summary of the findings from Tables 2-6 is as follows:

i) The Bayes estimates for both θ and λ demonstrate convergence to their nominal values.

ii) The rate parameter tends to be overestimated for both symmetric and asymmetric loss functions, while it underestimates the transmuted parameter when λ=1.

iii) The MLEs for the rate parameter exhibit a decreasing trend, whereas the MLEs for the transmuted parameter display an irregular pattern (as shown in Table 2).

iv) Variations are observed when comparing the Bayes estimates to the MLEs. Specifically, the Bayes estimates for the transmuted parameter when λ<0 under the PLF significantly exceed the MLEs.

v) As the sample size (n) increases, the MSE for the rate and transmuted parameters of the MLE decreases.

vi) Higher sample sizes (n) correspond to smaller posterior risks.

vii) Under the SELF, the combination of θ=0.5 and λ=-0.9 yields the best posterior risk.

viii) In the case of the PLF, posterior risk values exceed 1 for negative values of λ.

ix) Notably, the magnitude of posterior risks consistently remains smaller when employing the LINEX (m=-1) compared to the SELF and PLF.

Based on the findings presented above, it becomes evident that with an increase in sample size, the LINEX loss function with m=-1 consistently yields the smallest posterior risk when compared to the other two loss functions.

5. Application

In this section, the researchers illustrate the practical applicability of the QTE distribution by applying the estimation methods discussed earlier to real-world data. The dataset used in this analysis concerns the fatigue fracture duration of Kevlar 373/epoxy samples that were exposed to a sustained 90% stress level until failure occurred. This data-set was originally obtained from Abdul-Moniem and Seham (2015).The presented model is compared with Exponential Distribution (ED) and Generalized Exponential Distribution (GED). The researchers provide summary statistics and evaluate the goodness of fit criteria for the data-set, as detailed in Owoloko et al. (2015). To ascertain the suitability of the Quadratic Transmuted Exponential distribution for modeling this data-set, a Kolmogorov-Smirnov (K-S) test is conducted. The computed K-S test statistic is reported as 0.0965, with a corresponding P-value of 0.4504. These results indicate that the data closely adheres to the QTED distribution at a 5% level of significance. Hence, QTED can be seen as a superior model for the given data. All the observations are presented in the following Table 7. The data summary is presented in Table 8. Table 9 illustrates the performance of the QTED. Table 10 showcases the Bayesian parameter estimates of θ and λ under the three loss functions, along with the corresponding posterior risk values and the MLEs.

Table 7: Life of Fatigue Fracture of Kevlar 373/Epoxy.

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566
0.6748	0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113
0.9120	0.9836	1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630
1.7746	1.8275	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316	1.9558	2.0048
2.0408	2.0903	2.1093	2.1330	2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045	3.4846	3.7433	3.7455	3.9143
4.8073	5.4005	5.4435	5.5295	6.5541	9.0960				

Table 8: Summary of the Data on Life of Fatigue Fracture of Kevlar 373/epoxy.

Min.	Q_1	Q_2	Q_3	Max	Mean	Variance	Skewness	Kurtosis
0.0251	0.9048	1.7360	2.2960	9.0960	1.9590	2.4774	1.9406	8.1608

Table 9: Performance of the QTED for the Fatigue Fracture of Kevlar 373/epoxy.

Distribution	Parameters	Estimates	LogLik.	AIC	BIC	KS	P-value
QTED	$\hat{\theta}$ $\hat{\lambda}$	0.7266 -0.8487	-121.5166	247.0331	251.6947	0.0965	0.4504
ED	$\hat{\theta}$	0.5104	-127.1143	256.2287	258.5593	0.1663	0.0263
GED	$\hat{\theta}$ $\hat{\lambda}$	1.7095 0.7028	-122.2436	248.4872	253.1487	0.0942	0.4803

Table 10: Maximum Likelihood and Bayesian Estimation under the Three Loss Functions and its Posterior Risk.

Parameter	MLE	SELF	PLF	LINEX(m=-1)	LINEX(m=+1)
θ	0.7266	0.7346 (0.0041)	0.7374 (0.0056)	0.7366 (0.0020)	0.7325 (0.0021)
λ	-0.8487	-0.7895 (0.0119)	0.7970 (3.1730)	-0.7840 (0.0055)	-0.7959 (0.0064)

In evaluating the performance of these loss functions, it becomes apparent that the LINEX (m=-1) consistently produces the smallest posterior risks compared to all the other loss functions under consideration. Employing the Bayes Estimate within the LINEX (m=-1) loss function, Fig.1 illustrates the Bayes' reliability pattern, while Fig.2 depicts the Bayes' hazard rate function. The Bayes' estimate of MTTF for the fatigue fracture life of Kevlar 373/epoxy is established to be 1.8898. This signifies that the Bayes' estimate of average lifespan of the Kevlar 373/epoxy material is 1.8898 units of time. Consequently, a protocol is devised to replace or renew the Kevlar 373/epoxy material every 1.8898 units of time, ensuring continuity before the current one reaches the point of failure.

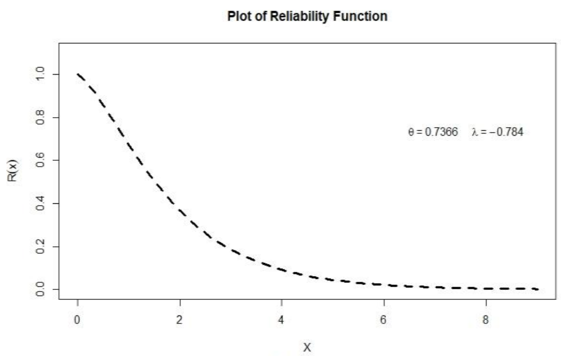

Fig.1

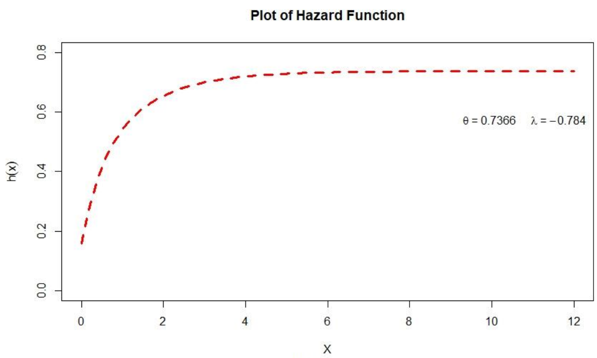

Fig.2

6. Conclusion

In response to the existing gap in understanding the performance of parameters within the Quadratic Transmuted Exponential Distribution while assuming non-informative priors, this research endeavors to address this gap comprehensively. The study encompasses an exploration of various symmetric and asymmetric loss functions to gain insights into the characteristics of the assumed distribution. The Lindley's approximation method is implemented to obtain Bayes estimators, while MLEs are derived using the Newton-Raphson method in RStudio. The selection of an appropriate loss function is a focal point, and to make this choice, an extensive Monte Carlo simulation study is conducted, encompassing diverse sample sizes and parameter values. Additionally, the research includes a practical example featuring a set of real-world values. A detailed examination of Tables 2 through 6 reveals that the Bayesian estimators consistently exhibit the property of consistency, wherein the Bayes estimates converge towards the nominal values.

Furthermore, it is observed that an increase in sample size corresponds to a decrease in posterior risks. The LINEX loss function consistently yields the minimum posterior risks among the various loss functions. Based on the minimum posterior risk obtained from the Bayes' estimates, the study delves into the Bayes' estimate of lifespan analysis of Kevlar 373/epoxy material. This analysis signifies that the Bayes' estimate of average lifespan of the Kevlar 373/epoxy material amounts to 1.8898 units of time. Consequently, a protocol is developed to replace or renew the Kevlar 373/epoxy material every 1.8898 units of time, ensuring continuity before the current material reaches its point of failure.

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