ISSN 0971-0388

Aligarh Journal of Statistics
Vol. 44(2024), 17-24

Bayes Estimators for the Parameters of Binomial Type Rayleigh Class Software Reliability Model Using Non-informative and Inverted Gamma Prior

Rajesh Singh[1], Kailash R. Kale[2] and Pritee Singh[3]
[Received on May, 2023. Accepted on January, 2024]

ABSTRACT

The paper deals with the cost-benefit analysis of a system composed of two non-identical units A and B arranged in series configuration with an identical cold standby unit corresponding to unit-A. System failure occurs when either unit-A (including its redundancy) or unit-B are in total failure mode. Failure and repair times of the units are independent random variables of discrete nature and follow geometric distributions with different parameters. The various system effectiveness measures are obtained using the regenerative point technique[1]. The reliability of software can be assessed by analyzing the pattern of failure occurrence at the software testing stage. To assess the reliability of Software, the Rayleigh Class software reliability growth model is proposed by assuming the Binomial pattern of failure occurrence. Also, the Bayes estimates for the number of inherent failures and the scale parameter are proposed using the non-informative, inverted gamma prior, respectively. The performance of proposed Bayes estimators are compared with corresponding MLEs based on relative efficiencies obtained by the Monte Carlo simulation technique for studying the performance of both proposed estimators.

1. Introduction

Today, computer and mobile applications have conquered the routine life of all human beings as well as various sectors of sciences and technology, engineering, humanities, and so on. Moreover, various software applications are used in the medical industry, healthcare industry, aviation industry, banking, government sectors, etc. These applications are the collective result of the many complex sequences of codes. Hence, there are high chances of induction of faults which leads to occurrence of failures or inefficient performance. These failures or inefficient software performance may be due to many reasons, such as errors in memory, language-specific errors, calling third-party libraries, standard libraries, etc. The operational effects of such failures may result in system breakdown and unexpected hazardous results. Therefore, many systems of the above sectors need reliable operations of applications/software. Therefore, the assessment and quantification of the performance of the software are very essential. One of the measures of the performance of the software is its reliability. In other words, it becomes essential to develop reliable software which satisfies the requirements of users or systems.

In this paper, the focus is on the quantification of reliability of software considering Binomial type failure occurrence having one parameter Rayleigh class failure intensity. The failure intensity is assumed to be a Rayleigh distribution involving parameters, such as the total number of inherent failures present in the software, i.e.,η_0 and the scale parameter, i.e. η_1. Many researchers use the Rayleigh Class failure intensity for quantifying the reliability of software c.f. Vouk (1992), Yamada *et al*. (1986), Kan (2002), Norden (1963), Putnam L. H. (1978). Schick and Wolverton (1973, 1978), Singh and Andure (2008), Singh and Kale (2016), Singh *et al.* (2016, 2022a, 2022b, 2023) etc.

Corresponding author : Rajesh Singh, Innovation Incubation and Linkages, Rashtrasant Tukdoji Maharaj Nagpur University, Nagpur, Maharashtra. Email:rsinghamt@gmail.com
[2]Department of Statistics, Ghulam Nabi Azad Arts, Commerce and Science College, Barshitakli, Dist-Akola. Maharashtra.
[3]Department of Statistics, Institute of Science, Nagpur, Maharashtra.

Suppose that no information or very less information is available about the parameter η_0 and it is likely that η_1 will be having an inverted gamma distribution as a priori. Here, the Bayes estimators are obtained for the Binomial type Rayleigh Class Software Reliability Growth Model considering non-informative and inverted gamma priors for the parameters η_0 and η_1 respectively (see Musa *et al.* (1984), Singh and Andure (2008), and Singh and Kale (2016)). The performance of proposed Bayes estimators is studied by comparing them with the corresponding maximum likelihood estimators. It is seen that the Bayesian estimators of η_0 and η_1 perform better than their corresponding MLEs for the small sample size.

2. Characterization of Model

Assumptions:

i) Failure occurrence process follows the Binomial Type

ii) The fault that causes failure will be removed immediately.

iii) There are finite inherent faults present in the software.

iv) Each failure occurs independently and randomly in time according to the constant per-fault hazard rate.

Let T be the non-negative, real number, random variable representing time to failure with realization t then Rayleigh Class failure intensity $\lambda(t) = \eta_0 f(t)$ and expected number of failures at time t_e i.e. $\eta_0 = \eta_0 F(t)$ is given by

$$\lambda(t) = \eta_0 t \eta_1^{-2} e^{-\frac{1}{2}[t\eta_1^{-1}]^2} \quad , \ t > 0, \eta_1 > 0, \eta_0 > 0 \tag{2.1}$$

and

$$\mu(t_e) = \eta_0 \left[1 - e^{-\frac{1}{2}(t_e \eta_1^{-1})^2}\right] \tag{2.2}$$

where

$$f(t) = t\eta_1^{-2} e^{-\frac{1}{2}[t\eta_1^{-1}]^2} \quad , \ t > 0, \eta_1 > 0, \eta_0 > 0$$

and

$$F(t_e) = \left[1 - e^{-\frac{1}{2}(t_e \eta_1^{-1})^2}\right]$$

If $M(t)$ is a random variable representing failures experienced in software up to time t then the probability of getting $M(t) = m$ failures is given as follows. $P[M(t) = m] = \binom{\eta_0}{m}\left[1 - e^{-\frac{1}{2}(t_e \eta_1^{-1})^2}\right]^m \left[e^{-\frac{1}{2}(t_e \eta_1^{-1})^2}\right]^{(\eta_0 - m)}, m = 0,1,2,\ldots,\eta_0$ (2.3)

This is known as the Binomial type model (c.f. Musa *et al.* (1984))

3. Maximum Likelihood Estimators

Suppose software is executed up to t_e time and if $t_1, t_2, \ldots, t_{m_e} (= t)$ be failure times of m_e failures experienced up to execution time t_e. In general, using $f(t)$ and $F(t_e)$ the likelihood function is

$$L(\eta_0, \eta_1 / \underline{t}) = [1 - F(t_e)]^{\eta_0 - m_e} \prod_{i=1}^{m_e} (\eta_0 - i + 1) f(t_i)$$

After substituting the values of $f(t)$ and $F(t_e)$ and doing mathematical simplification, the above likelihood function becomes

$$L(\eta_0, \eta_1) = e^{-\left[0.5(t_e\eta_1^{-1})^2(\eta_0 - m_e)\right]} \eta_1^{-2m_e} \left[\prod_{i=1}^{m_e} t_i\right] e^{-0.5T\eta_1^{-2}} \eta_0^{\underline{m_e}} \tag{3.1}$$

where,

$\eta_0^{\underline{m_e}} = \prod_{i=1}^{m_e}(\eta_0 - i + 1)$ is falling factorials (cf. Gradshteyn and Ryzhik (1994), Graham et al. (1994) and Osgood and Wu (2009)).

The maximum likelihood estimators of parameters η_0 and η_1 can be obtained by differentiating (3.1) with respect to η_0 and η_1 and equating with zero; then, after simplification, we get

$$\sum_{i=1}^{m_e}(\hat{\eta}_{m0} - i + 1)^{-1} = 0.5(t_e\eta_1^{-1})^2 \tag{3.2}$$

and

$$(\hat{\eta}_{m1})^2 = 0.5[(\hat{\eta}_{m0} - m_e)t_e^2 + T]m_e^{-1} \tag{3.3}$$

The MLEs $\hat{\eta}_{m0}$ and $\hat{\eta}_{m1}$ can be obtained by solving the above equations by using any standard iteration method.

4. Priors

Let us assume that the experimenter has very little information about the parameter η_0 or the information about the parameter η_0 is not available up to the software testing to the experimenter. Hence the non-informative prior to the number of failures η_0 will be a correct selection, i.e.

$$g(\eta_0) \propto \begin{cases} \eta_0^{-1} & ,\eta_0 \epsilon\ [0,\infty) \\ 0 & ,\text{Otherwise} \end{cases}$$

Suppose the prior information about the scale parameter η_1 available to the researcher, and it is guessed to follow an inverted gamma distribution. Then, the prior for η_1 can be selected as

$$g(\eta_1) = \begin{cases} {\eta_1}^{-\alpha-1}e^{-\beta{\eta_1}^{-1}} & ,\alpha > 0, \beta > 0, 0 < \eta_1 < \infty \\ 0 & ,\text{Otherwise} \end{cases}$$

Assuming that η_0 and η_1 are independent, the joint prior distribution of η_0 and η_1 can be written by multiplying both priors as

$$g(\eta_0,\eta_1) = \begin{cases} {\eta_1}^{-\alpha-1}e^{-\beta{\eta_1}^{-1}}{\eta_0}^{-1} & ,\alpha > 0, \beta > 0, 0 < (\eta_0,\eta_1) < \infty \\ 0 & ,\text{Otherwise} \end{cases} \tag{4.1}$$

where α and β are prior constants of η_1.

5. Bayesian Estimation

Joint Posterior of $\boldsymbol{\eta_0}$ and $\boldsymbol{\eta_1}$:

The joint posterior of η_0 and η_1 given $\underline{t}$ is obtained with the help of equations (3.1) and (4.1) by applying Bayes theorem, which comes out as

$$\pi(\eta_0,\eta_1|\underline{t}) = D^{-1}e^{-\sqrt{2}[t_e(\eta_0-m_e)+T+\beta]\eta_1^{-1}}\eta_1^{-2m_e-\alpha-1}\eta_0\underline{^{m_e}}{\eta_0}^{-1} \tag{5.1}$$

$,m_e < \eta_0 < \infty, 0 < \eta_1 < \infty$ and $0 < T < t_e$

The constant of proportionality D is defined as

$$D = C_3\sum_{m=0}^{m_e}S_{m_e}^{(m)}[T_2]^m\sum_{k=0}^{m-1}\frac{(m-1)!}{k!}[T_3]^k\Gamma(C_2)$$

where

$C_1 = 2m_e + \alpha,$

$C_2 = 2m_e + \alpha + k - m,$

$C_3 = \left[\sqrt{2}(T+\beta)\right]^{-C_1},$

$T_2 = (T+\beta)t_e^{-1}$, and

$T_3 = t_e m_e(T+\beta)^{-1}$

Marginal of $\boldsymbol{\eta_0}$ and $\boldsymbol{\eta_1}$:

The marginal posterior distributions of the parameters η_0 and η_1 are obtained by integrating other parameters over the whole range and denoted by $\pi(\eta_0|\underline{t})$ and $\pi(\eta_1|\underline{t})$ respectively. Thus,

$$\pi(\eta_0|\underline{t}) = D^{-1}\eta_0\underline{^{m_e}}{\eta_0}^{-1}\left\{\sqrt{2}[t_e(\eta_0 - m_e) + T + \beta]\right\}^{-C_1}\Gamma(C_1) \tag{5.2}$$

$,m_e < \eta_0 < \infty$ and $0 < T < t_e$

and

$$\pi(\eta_1|\underline{t}) = D^{-1}e^{-\sqrt{2}[T+\beta]\eta_1^{-1}}\eta_1^{-C_1-1}\, S(m_e, m, k, t_e, \eta_1) \tag{5.3}$$
$0 < \eta_1 < \infty$ and $0 < T < t_e$

where

$$S(m_e, m, k, t_e, \eta_1) = \sum_{m=0}^{m_e} S_{m_e}^{(m)} \sum_{k=0}^{m-1}(m-1)!\,(k!)^{-1} {m_e}^{k}(\sqrt{2}t_e\eta_1^{-1})^{-(m-k)}$$

Proposed Bayes Estimates of η_0 and η_1:

The proposed Bayes estimator $\hat{\eta}_{B0}$ is the posterior mean and can be obtained from (5.2), which is

$$\hat{\eta}_{B0} = T_2 D_1 \sum_{m=0}^{m_e} S_{m_e}^{(m)}\, [T_2]^m \sum_{k=0}^{m} m!\,(k!)^{-1}\, [T_3]^k \Gamma(C_4)$$

$0 < T < t_e$, $\alpha > 0$ and $\beta > 0$ (5.4)

where

$$D_1 = \left[\sum_{m=0}^{m_e} S_{m_e}^{(m)}\, [T_2]^m \sum_{k=0}^{m-1}(m-1)!\,(k!)^{-1}\, [T_3]^k \Gamma(C_2)\right]^{-1}$$

and

$$C_4 = 2m_e + \alpha - m + k - 1$$

The proposed Bayes estimator $\hat{\eta}_{B1}$ is the posterior mean and can be obtained from (5.3) as

$$\hat{\eta}_{B1} = C_5 D_2 \sum_{m=0}^{m_e} S_{m_e}^{(m)}\, [T_2]^m \sum_{k=0}^{m-1} \frac{(m-1)!}{k!}\, [T_3]^k \Gamma(C_4) \tag{5.5}$$

where

$$D_2 = \left[\sum_{m=0}^{m_e} S_{m_e}^{(m)}\, [T_2]^m \sum_{k=0}^{m-1} \frac{(m-1)!}{k!}\, [T_3]^k \Gamma(C_2)\right]^{-1}, \text{ and}$$

$$C_5 = \sqrt{2}[T + \beta]$$

6. Result and Discussion

The Bayes estimators are proposed by considering inverted gamma prior to scale parameter η_1 and non-informative prior for parameter η_0 in section 5.3 i.e. $\hat{\eta}_{B1}$ and $\hat{\eta}_{B0}$ and proposed estimators are compared with corresponding MLEs obtained in section 3. The proposed Bayes estimators are depending upon the values of prior constants α, β, and execution time t_e. Therefore, the performances of proposed Bayes estimators are studied by taking different values of prior constants α, β and execution time t_e and different hypothetical true values of parameters η_0 and η_1. This study is carried out by evaluating the risk efficiencies for a sample generated up to t_e using the Inverse Transformation method of sample generation. The failure times t_i, $i = 1,2, \dots, m_e$, obtained by generating a random sample up to a fixed execution time t_e. The risk efficiencies were evaluated by repeating the process 103 times, generating samples using the Monte Carlo Simulation technique. The performance of proposed Bayes estimators is summarized and presented in the form of graphs G-1 to G-6 of risk efficiencies RE_0 and RE_1 obtained by taking $t_e(= 4.0(2.0)8.0)$, $\alpha(= 1,10,20)$, $\beta(= 5(5)15)$, $\eta_0(= 30(1)39)$ and $\eta_1(= 1.0(0.5)5.5)$.

The region in which Bayes estimators perform better than MLEs is obtained from Graphs G-1 to G-6 and summarized in Table T-1.

t_e	α	β	Region for efficient $\hat{\eta}_{B0}$	Region for efficient $\hat{\eta}_{B1}$
4.0	10	5	$\forall\eta_0$ and η_1	$\forall\eta_0$ and η_1
6.0	10	5	$\forall\eta_0$ and η_1	$\forall\eta_0$ and η_1
8.0	10	5	$\forall\eta_0$ and η_1	$\forall\eta_0$ and η_1
6.0	1	5	$\forall\eta_0$ and η_1	$\forall\eta_0$ and $\eta_1 > 1.0$
6.0	20	5	$\forall\eta_0$ and η_1	$\forall\eta_0$ and η_1
6.0	10	10	$\forall\eta_0$ and η_1	$\forall\eta_0$ and η_1
6.0	10	15	$\forall\eta_0$ and η_1	$\forall\eta_0$ and $\eta_1 > 1.0$

Table (T- 1): Behavior of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$ for different values of Execution Time and hyperparameters.

7. Conclusion

Both the proposed Bayes estimators of η_0 and η_1 i.e. $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$ can be preferred over corresponding MLEs if the uniform prior is suitable for η_0 and inverted gamma prior for η_1. For the large execution time t_e, the proposed estimator $\hat{\eta}_{B0}$ performs well for larger values of η_1 and smaller values of η_0. Similarly, $\hat{\eta}_{B1}$ performs better than MLE for all values of η_0 and η_1 except a few cases.

References

Abramowitz, M. and Stegun, I. A. (1965): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover publications, New York.

Ana Maria VLADU (2011): Software Reliability Prediction Model Using Rayleigh Function. *U.P.B. Sci. Bull., Series C*, **73(4)**, 55-64

Gradshteyn, I. S. and Ryzhik, I. M. (1994): Table of Integrals, Series, and Products, Alan Jeffrey (editor) 5th Ed., Academic Press, New York.

Graham, R. L. Knuth, D. E. and Patashnik, O. (1994): Concrete Mathematics: A Foundation for Computer Science, 2nd Ed. New York: Addison-Wesley Publishing Co.

Kan, S. H. (2002): Metrics and Models in Software Quality Engineering. 2nd Ed., Addison Wesley, New York.

Lyu, M. R. (2007): Software Reliability Engineering: A Roadmap, Future of Software Engineering, *(FOSE '07), 2007*, 153-170.

Musa, J. D., Iannino, A. and Okumoto, K. (1987): Software Reliability: Measurement, Prediction, Application. McGraw-Hill, New York.

Norden, P. V., (1963): Useful Tools for Project Management, Operations Research in Research and Development, B. V. Dean, Ed., John Wiley and Sons, New York.

Osgood, B. and Wu, W. (2009): Falling factorials, generating functions and conjoint ranking tables, *J. Int. Seq.*, **12**, Article 09.7.8.

Putnam L. H. (1978): A general empirical solution to the macro software sizing and estimating problem, *IEEE Trans. Soft. Engi.*, **4**, 345–367.

Schick G. J. and Wolverton R. W. (1973). Assessment of software reliability, *Proc. of Operations Research*, *Physica-Verlag,Wurzburg-Wien*, 395-422.

Schick G. J. and Wolverton R. W. (1978): An analysis of competing software reliability models, *IEEE Trans. Soft. Engi*., **4(2)**, 104-120.

Singh R. and Andure N. W. (2008): Bayes estimators for the parameters of the Poisson type exponential distribution, *IAPQR trans.,* **33 (2)**, 121-128.

Singh R. and Kale K. R. (2016): Bayes estimates for the parameters of Poisson type length biased exponential class model using non-informative priors, *Journal of Reliability and Statistical Studies*; **9(1)**, 21-28.

Singh R., Singh P. and Kale K. R. (2016): Bayes Estimators of Parameters of Binomial Type Rayleigh Class Software Reliability Growth Model Using Non-Informative Priors. *Journal of Reliability and Statistical Studies*; **11(2)**, 01-07.

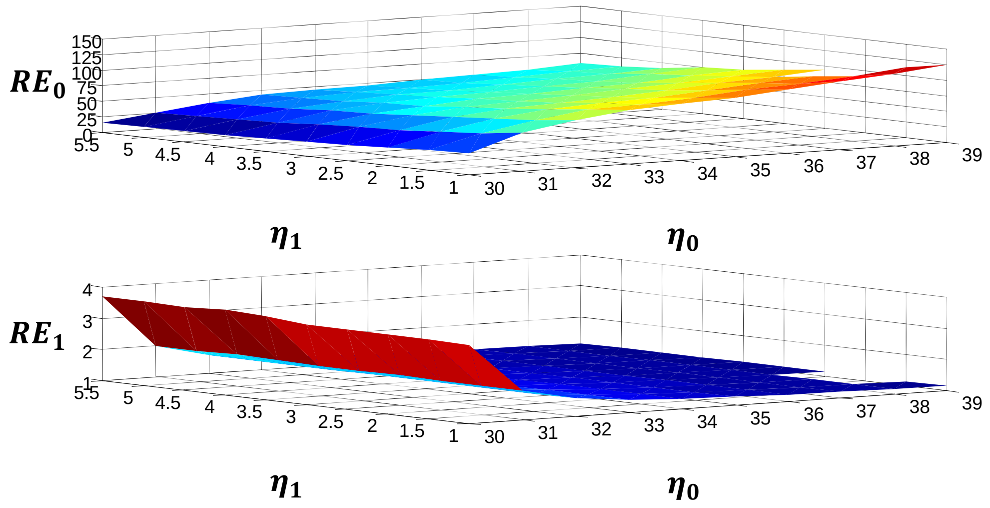
Singh R., Singh P. and Kale K. R. (2022a): Estimation of Parameters of PTRC SRGM using Non-informative Priors. *International Journal of Advanced Research in Science, Communication and Technology*, **2(1),** 172-178.

Singh R., Badge P., and Singh P. (2022b): Bayesian Interval Estimation for the Parameters of Poisson Type Rayleigh Class Model. *Reliability: Theory & Applications,* **17(4),** 98-108.

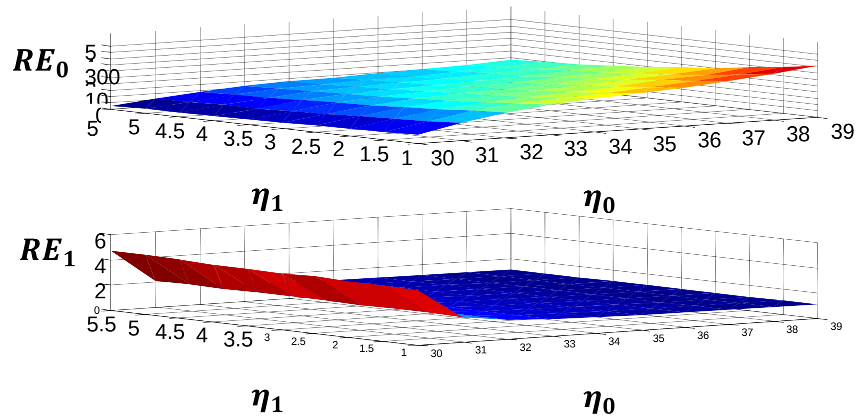
Singh R., Badge P., and Singh P. (2023): Bayesian Interval Estimation for the Parameters of Poisson Type Length Biased Exponential Class Model. *Reliability: Theory & Applications*, **18(2)**, p. 307-314.

Vouk M. A. (1992): Using reliability models during testing with non- operational profile, *Proc. 2nd Bellcore/Purdue Symp. Issues in Software Reliability Estimation*, 103–110.

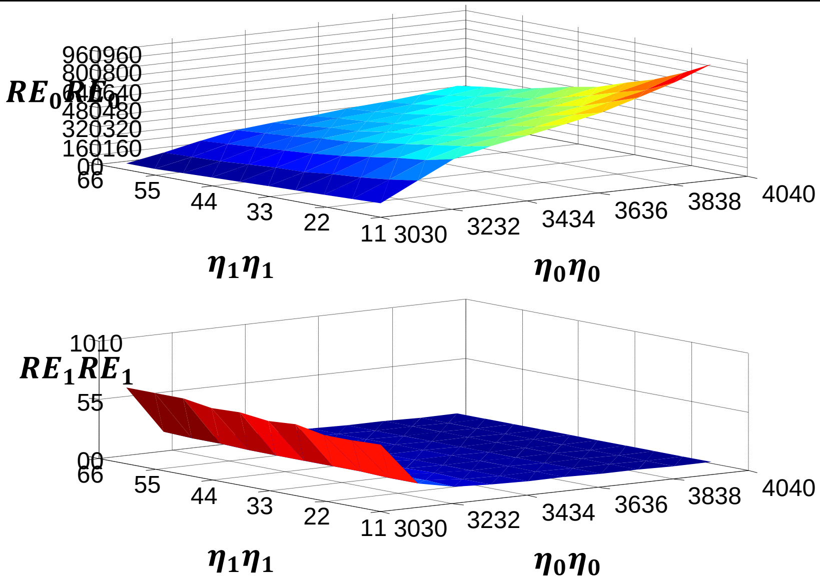
Yamada, S. Ohtera, H. and Narihisa, H. (1986): Software reliability growth models with testing-effort, *IEEE Trans. Reliability*, **35**, 19 – 23.



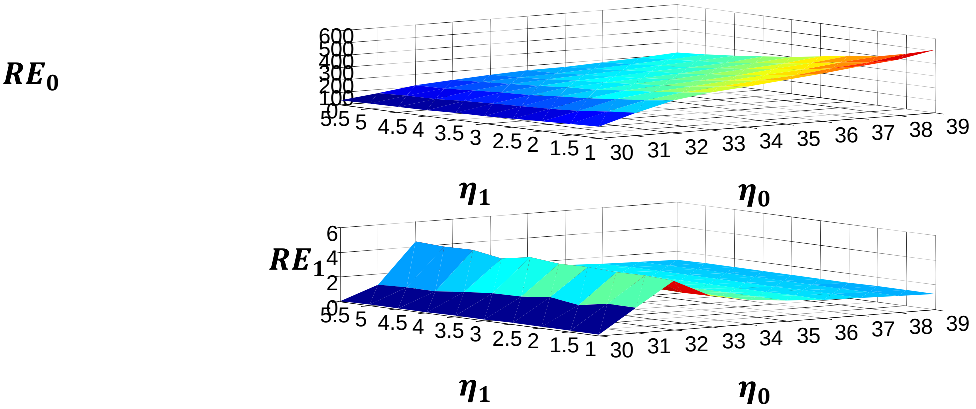
Graph (G-1): Risk Efficiencies of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$, for $\eta_1 = 1.0(0.5)5.5$, $\eta_0 = 30(1)39$, $\alpha = 10$, $\beta = 5$ and $t_e = 4.0$.



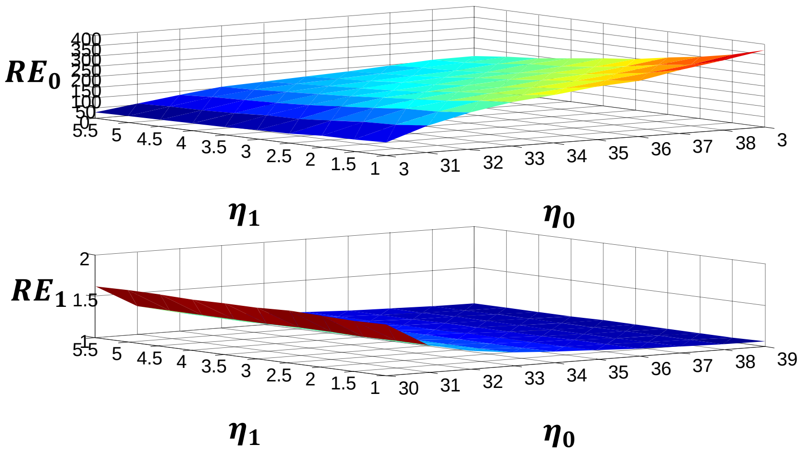
Graph (G-2): Risk Efficiencies of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$, for $\eta_1 = 1.0(0.5)5.5$, $\eta_0 = 30(1)39$, $\alpha = 10$, $\beta = 5$ and $t_e = 6.0$.



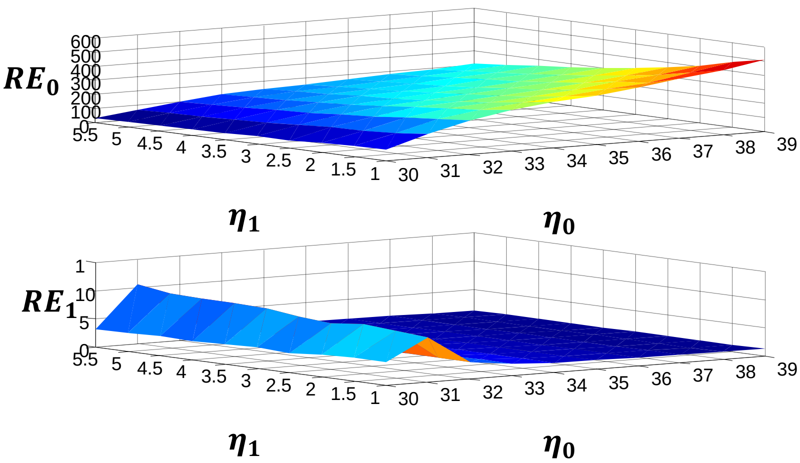
Graph (G-3): Risk Efficiencies of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$, for $\eta_1 = 1.0(0.5)5.5$, $\eta_0 = 30(1)39$, $\alpha = 10$, $\beta = 5$ and $t_e = 8.0$.



Graph (G-4): Risk Efficiencies of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$, for $\eta_1 = 1.0(0.5)5.5$, $\eta_0 = 30(1)39$, $t_e = 6$, $\beta = 5$ and $\alpha = 1.0$.



Graph (G-5): Risk Efficiencies of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$, for $\eta_1 = 1.0(0.5)5.5$, $\eta_0 = 30(1)39$, $t_e = 6$, $\beta = 5$ and $\alpha = 20$.



Graph (G-6): Risk Efficiencies of $\hat{\eta}_{B0}$ and $\hat{\eta}_{B1}$, for $\eta_1 = 1.0(0.5)5.5$, $\eta_0 = 30(1)39$, $t_e = 6$, $\alpha = 10$ and $\beta = 10$.