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Three Parameter Bivariate Weighted Geometric Distribution

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ABSTRACT

The main aim of this work is to introduce a new three parameter bivariate weighted geometric distribution. In this article, we derived certain properties of the proposed distribution. We used different methods to estimate the unknown parameters to assess the performance of the derived bivariate distribution. We used two real life data sets to demonstrate the applicability of the model. Simulation study is also carried out.

1. Introduction

Getting samples from a continuous distribution can be difficult or inconvenient in real life situations. Because observed values are usually measured to only a finite number of decimal places, they cannot really represent all points in a continuum. According to Chakraborty (2015), measurements taken on a continuous scale may be recorded to make discrete models more appropriate. Deriving discrete analogues of continuous distributions has attracted researchers' attention. Different discrete distributions have been derived by discretizing known continuous distributions. In practice, discrete bivariate data are often analyzed. Many real-life situations involve discrete bivariate data.

As a discrete analogue of Gumblels' bivariate exponential distribution, Nair and Nair (1988) defined a bivariate geometric distribution. Using ideas from Freund (1961) reliability models, Dhar (1998) developed another bivariate geometric distribution. Many authors have described a variety of bivariate geometric distributions, providing additional insight into these models. Basu and Dhar (1995) generalized the multivariate version of bivariate geometric distribution by Sun and Basu (1995). With probability parameters strictly between 0 and 1, this multivariate geometric distribution is the discrete analogue of Marshall and Olkin (1967)'s multivariate exponential distribution. Bhati and Joshi (2018) introduced a weighted geometric distribution, and Najarzadegan and Alamatsaz (2017) generalized the weighted geometric distribution based on Azzalini (1985). The weighted geometric distribution introduced by Bhati and Joshi (2018) was a discrete analogue of the weighted exponential distribution proposed by Gupta and Kundu (2009). In addition, Najarzadegan *et al.* (2018) introduced a new weighted bivariate geometric distribution.

A new discrete weighted exponential distribution was developed by Khongthip *et al.* (2018). In literature, weighted bivariate continuous distributions are available. Jamalizadeh and Kundu (2013) introduced the weighted Marshall-Olkin bivariate exponential distribution. Al-Mutairi et al. (2011) introduced a new bivariate distribution with weighted exponential marginals, and its multivariate generalization is also presented. Using the Weighted Weibull Distribution, Al-Mutairi *et al.* (2018) extended the bivariate model to multivariate data. The main aim of this work is to construct a weighted geometric distribution, which can be taken as a discrete analogue of a special case of Ghosh

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and Alzaatreh (2016). Our proposed model is more flexible than the existing geometric and weighted geometric distributions. It has applications in risk and shock models.

The rest of this study is arranged as follows. Section 2 discusses the preliminary details, which include weighted geometric distribution, bivariate geometric distribution, weighted bivariate geometric distribution and bivariate weighted exponential distribution. Section 3 discusses the properties of three parameter bivariate weighted geometric distribution. Section 4 provides a new bivariate weighted geometric distribution, the definition of the proposed distribution and its properties such as joint cumulative distribution function, joint probability mass function, moment generating function, conditional distributions, moments, characteristics function, maximum likelihood estimation etc. Also, in section 5 simulation study is done. Section 6 and 7 discuss the estimation and real data analysis, respectively. Section 8 concludes the work.

Here are some preliminary considerations.

2. Preliminaries

a) Bivariate Weighted Generalized Exponential Distribution

Ghosh and Alzaatreh (2016) proposed a bivariate generalized exponential distribution with joint probability mass function.

$$f(x,y) = \theta^{-2}\left(\frac{1}{\alpha_0+\alpha_1+\alpha_2}\left(\frac{1}{\alpha_0+\alpha_1}+\frac{1}{\alpha_0+\alpha_2}\right)\right)^{-1} e^{-\left(\frac{x}{\theta}+\frac{y}{\theta}\right)}\left(1-e^{-\frac{x}{\theta}}\right)^{\alpha_1-1}\left(1-e^{-\frac{y}{\theta}}\right)^{\alpha_2-1}\left(1-e^{-\frac{\min(x,y)}{\theta}}\right)^{\alpha_0}$$

where x > 0; y > 0 and $\alpha_i > 0$; i = 0, 1, 2 and θ > 0 are the parameters of the distribution.

b) Bivariate Geometric Distribution

If U_1, U_2 and U_3 are geometric distributions with parameters 1-q_1, 1-q_2 and 1-q_3 respectively then (X,Y)$\overset{d}{=}(min(U_1,U_3), min(U_2,U_3))$ is a bivariate geometric distribution (BGE) introduced by Basu and Dhar (1995). The joint probability mass function of (X, Y) is given by

$$f_{BGE}(x,y) = \begin{cases} K_1 {q_1}^x(1-q_1)(q_2q_3)^y(1-q_2q_3), & if\ \ x<y \\ K_1 {q_2}^y(1-q_2)(q_1q_3)^x(1-q_1q_3), & if\ x>y \\ K_1(q_1q_2q_3)^x(1-q_1q_3-q_2q_3+q_1q_2q_3), & if\ \ x=y, \end{cases}$$

Where ${K_1}^{-1} = q_1q_2q_{3,}$ $x \in \mathrm{N}_0, y \in \mathrm{N}_0$ and q_1, q_2 and q_3, are parameters of the distribution.

c) Weighted Bivariate Geometric Distribution

If (X_1, X_2) is bivariate geometric distribution (BGE) introduced by Basu and Dhar

(1995) and X_3 is geometric distribution with parameter $1-q_4$, then $(X,Y)\overset{d}{=}(X_1,X_2)/X_3 < min(X_1,X_2)$ is a weighted bivariate geometric distribution (WGBE) proposed by Najarzadegan *et al.* (2018). The random variable WBGE with parameters q_1, q_2, q_3 and q_4 has joint probability mass function, which is given by

$$f_{WBGE}(x,y) = \begin{cases} K_2 {q_1}^x(1-q_1)(q_2q_3)^y(1-q_2q_3)(1-{q_4}^{x+1}), & if\ \ x<y \\ K_2 {q_2}^y(1-q_2)(q_1q_3)^x(1-q_1q_3)(1-{q_4}^{y+1}), & if\ \ x>y \\ K_2(q_1q_2q_3)^x(1-q_1q_3-q_2q_3+q_1q_2q_3)(1-{q_4}^{x+1}), & if\ \ x=y, \end{cases}$$

Where ${K_2}^{-1} = \frac{1-q_4}{1-q_1q_2q_3q_4}$, x$\epsilon \mathrm{N}_{0,}$ y$\epsilon \mathrm{N}_0$.

d) Bivariate Weighted Exponential Distribution

Al-Mutairi *et al.* (2011) introduced a bivariate distribution with parameters $\lambda_1, \lambda_2, \lambda_3$ which has joint cumulative distribution function as

$$F(x,y) = (1-e^{-\lambda z}) - \frac{\lambda}{\lambda_1+\lambda_3} e^{-\lambda_2 y}(1-e^{-(\lambda_1+\lambda_3)z}) - \frac{\lambda}{\lambda_2+\lambda_3} e^{-\lambda_1 x}(1-e^{-(\lambda_2+\lambda_3)z}) + \frac{\lambda}{\lambda_3} e^{-\lambda_1 x} e^{-\lambda_2 y}(1-e^{-\lambda_3 z})),$$

where $z = min(x,y)$ and $\lambda = \lambda_1+\lambda_2+\lambda_3$.

3. Three Parameter Bivariate Weighted Geometric Distribution

Let U_1, U_2 and U_3 be three independent geometric random variables with respective probability mass functions and distribution functions,

$$f_{U_i}(u) = {p_i}^u - {p_i}^{u+1}, i = 1,2,3, u = 0,1,2, \dots$$

and

$$F_{U_i}(u) = 1 - {p_i}^{u+1}, i = 1,2,3.$$

Let $(X,Y) \stackrel{d}{=} (U_1,U_2)/U_3 < min(U_1,U_2)$. Then (X,Y) follows a bivariate weighted geometric distribution with parameters p_1, p_2 and p_3, which is denoted by $BWGD(p_1,p_2,p_3)$.

a) The Joint Probability Mass Function

Let $(X,Y) \sim BWGD(p_1,p_2,p_3)$; then the joint probability mass function of (X,Y) is

$$f_{BWGD}(x,y) = {p_1}^{x-1}{p_2}^{y-1}(1-p_1)(1-p_2)(1-p_1p_2p_3)\frac{1-{p_3}^z}{1-p_3}$$

where $z = min(x,y)$.

Proof:

By definition, we have

$$f_{BWGD}(x,y) = \frac{P(U_1=x)P(U_2=y)P(U_3 \le min(U_1,U_2)/U_1=x,U_2=y)}{\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} P(U_1=x)P(U_2=y)P(U_3 \le min(U_1,U_2)/U_1=x,U_2=y)}$$

where

$P(U_1=x)P(U_2=y)P(U_3 \le min(U_1,U_2)/U_1=x,U_2=y) = {p_1}^x{p_2}^y(1-p_1)(1-p_2)(1-{p_3}^{min(x,y)})$

and the normalizing constant,

$$K^{-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}({p_1}^i - {p_1}^{i+1})({p_2}^j - {p_2}^{j+1})(1-{p_3}^{min(i,j)})$$

$$= \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} {p_1}^i{p_2}^j(1-p_1)(1-p_2)(1-{p_3}^i)$$

$$+\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} {p_1}^i{p_2}^j(1-p_1)(1-p_2)(1-{p_3}^j)$$

$$+\sum_{i=0}^{\infty}(p_1p_2)^i(1-p_1)(1-p_2)(1-{p_3}^i).$$

On simplification, we get

$$K^{-1} = \frac{p_1 p_2 (1-p_3)}{1-p_1 p_2 p_3}.$$

Thus,

$$f_{BWGD}(x,y) = {p_1}^{x-1}{p_2}^{y-1}(1-p_1)(1-p_2)(1-p_1p_2p_3)\frac{1-{p_3}^z}{1-p_3},$$

where $z = min(x,y)$.

b) The Joint Distribution Function

If $(X,Y) \sim BWGD(p_1,p_2,p_3)$, then the joint distribution function of (X, Y) is given by

$$F_{BWGD}(x,y) = 1-(p_1p_2p_3)^{z+1} - {p_2}^y(1-p_1p_2p_3)\frac{1-(p_1p_3)^{z+1}}{1-p_1p_3} - {p_1}^x(1-p_1p_2p_3)\frac{1-(p_2p_3)^{z+1}}{1-p_2p_3} + {p_1}^x{p_2}^y(1-p_1p_2p_3)\frac{1-{p_3}^{z+1}}{1-p_3},$$

where $z = min(x,y)$.

Proof:

We have $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$$= P\big(U_1 \le x, U_2 \le y / U_3 < min(U_1,U_2)\big) = \frac{P(U_3 < U_1 \le x, U_3 < U_2 \le y)}{P(U_1 > U_3, U_2 > U_3)}.$$

Consider

$$P(U_3 < U_1 \le x, U_3 < U_2 \le y) = \sum_{u=0}^{z}({p_1}^{u+1} - {p_1}^{x+1})({p_2}^{u+1} - {p_2}^{y+1})({p_3}^{u-1} - {p_3}^{u})$$

$$= \frac{p_1p_2(1-p_3)}{p_3}\left[\frac{1-(p_1p_2p_3)^{z+1}}{1-p_1p_2p_3} - {p_2}^y\frac{1-(p_1p_3)^{z+1}}{1-p_1p_3} - {p_1}^x\frac{1-(p_2p_3)^{z+1}}{1-p_2p_3} + {p_1}^x{p_2}^y\frac{1-{p_3}^{z+1}}{1-p_3}\right]$$

where $z = min(x,y)$. Since $min(U_1,U_2)$ is a geometric distribution with parameter p_1p_2,

$$P(X_1 > X_3, X_2 > X_3) = \sum_{u=0}^{\infty}(p_1p_2)^{u+1}({p_3}^{u-1} - {p_3}^{u}) = \frac{p_1p_2(1-p_3)}{p_3}\sum_{u=0}^{\infty}(p_1p_2p_3)^u$$

$$= \frac{p_1p_2(1-p_3)}{p_3(1-p_1p_2p_3)}.$$

Thus, we get

$$F_{BWGD}(x,y) = 1-(p_1p_2p_3)^{z+1} - {p_2}^y(1-p_1p_2p_3)\frac{1-(p_1p_3)^{z+1}}{1-p_1p_3} - {p_1}^x(1-p_1p_2p_3)\frac{1-(p_2p_3)^{z+1}}{1-p_2p_3} + {p_1}^x{p_2}^y(1-p_1p_2p_3)\frac{1-{p_3}^{z+1}}{1-p_3},$$

where $z = min(x,y)$.

c) Moment Generating Function

Let $t_1 < -ln(p_1), t_2 < -ln(p_2)$ and $t_1 + t_2 < -ln(p_1 p_2 p_3)$. The moment generating function of $BWGD(p_1, p_2, p_3)$ is given by

$$M_{X,Y}(t_1, t_2) = \frac{(1-p_1)(1-p_2)(1-p_1p_2p_3)e^{t_1+t_2}}{(1-p_1e^{t_1})(1-p_2e^{t_2})(1-p_1p_2p_3e^{t_1+t_2})}.$$

Proof:

We have

$$M_{X,Y}(t_1, t_2) = E(e^{t_1X+t_2Y})$$

$$= \sum_{x=0}^{\infty}\sum_{y=x+1}^{\infty} e^{t_1x+t_2y}\, {p_1}^{x-1}{p_2}^{y-1}(1-p_1)(1-p_2)(1-p_1p_2p_3)\frac{1-{p_3}^x}{1-p_3}$$

$$+\sum_{y=0}^{\infty}\sum_{x=y+1}^{\infty} e^{t_1x+t_2y}\, {p_1}^{x-1}{p_2}^{y-1}(1-p_1)(1-p_2)(1-p_1p_2p_3)\frac{1-{p_3}^y}{1-p_3}$$

$$+\sum_{x=0}^{\infty} e^{(t_1+t_2)x}\,(p_1p_2)^{x-1}(1-p_1)(1-p_2)(1-p_1p_2p_3)\frac{1-{p_3}^x}{1-p_3}$$

$$= \frac{(1-p_1)(1-p_2)(1-p_1p_2p_3)}{p_1p_2(1-p_3)}\left[\sum_{x=0}^{\infty}\sum_{y=x+1}^{\infty}(p_1e^{t_1})^x\,(p_2e^{t_2})^y(1-{p_3}^x)\right.$$

$$\left.+\sum_{y=0}^{\infty}\sum_{x=y+1}^{\infty}(p_1e^{t_1})^x\,(p_2e^{t_2})^y(1-{p_3}^y)+\sum_{x=0}^{\infty}(p_1p_2e^{t_1+t_2})^x(1-{p_3}^x)\right]$$

$$= \frac{(1-p_1)(1-p_2)(1-p_1p_2p_3)}{p_1p_2(1-p_3)}\left[\sum_{x=0}^{\infty}(p_1e^{t_1})^x(1-{p_3}^x)\frac{(p_2e^{t_2})^{x+1}}{1-p_2e^{t_2}}\right.$$

$$+\sum_{y=0}^{\infty}(p_2e^{t_2})^y(1-{p_3}^y)\frac{(p_1e^{t_1})^{y+1}}{1-p_1e^{t_1}}$$

$$\left.+\sum_{x=0}^{\infty}(p_1p_2e^{t_1+t_2})^x(1-{p_3}^x)\right]$$

$$= \frac{(1-p_1)(1-p_2)(1-p_1p_2p_3)}{p_1p_2(1-p_3)}\left[\frac{p_2e^{t_2}}{1-p_2e^{t_2}}[(1-p_1p_2e^{t_1+t_2})^{-1}-(1-p_1p_2p_3e^{t_1+t_2})^{-1}]\right.$$

$$+\frac{p_1e^{t_1}}{1-p_1e^{t_1}}[(1-p_1p_2e^{t_1+t_2})^{-1}-(1-p_1p_2p_3e^{t_1+t_2})^{-1}]$$

$$\left.+[(1-p_1p_2e^{t_1+t_2})^{-1}-(1-p_1p_2p_3e^{t_1+t_2})^{-1}]\right]$$

$$= \frac{p_1 p_2 (1-p_1)(1-p_2)(1-p_3)(1-p_1 p_2 p_3) e^{t_1+t_2}}{p_1 p_2 (1-p_3)(1-p_1 p_2 e^{t_1+t_2})(1-p_1 p_2 p_3 e^{t_1+t_2})} \left[1 + \frac{p_1 e^{t_1}}{1-p_1 e^{t_1}} + \frac{p_2 e^{t_2}}{1-p_2 e^{t_2}}\right]$$

$$= \frac{(1-p_1)(1-p_2)(1-p_1 p_2 p_3) e^{t_1+t_2}}{(1-p_1 e^{t_1})(1-p_2 e^{t_2})(1-p_1 p_2 p_3 e^{t_1+t_2})},$$

where $t_1 < -ln(p_1), t_2 < -ln(p_2)$ and $t_1 + t_2 < -ln(p_1 p_2 p_3)$.

Theorem 1:

If (X,Y) is a three parameter bivariate weighted geometric distribution,
then $X \sim WG\left(\frac{lnp_2 p_3}{lnp_1}, p_1\right)$ and $Y \sim WG\left(\frac{lnp_1 p_3}{lnp_2}, p_2\right)$.

Proof:

We have

$$M_{X,Y}(t_1, t_2) = \frac{(1-p_1)(1-p_2)(1-p_1 p_2 p_3) e^{t_1+t_2}}{(1-p_1 e^{t_1})(1-p_2 e^{t_2})(1-p_1 p_2 p_3 e^{t_1+t_2})}.$$

$$M_X(t_1) = M_{X,Y}(t_1, 0) = \frac{(1-p_1)(1-p_1 p_2 p_3) e^{t_1}}{(1-p_1 e^{t_1})(1-p_1 p_2 p_3 e^{t_1})}$$

which is the MGF of $X \sim WG\left(\frac{lnp_2 p_3}{lnp_1}, p_1\right)$. Similarly

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{(1-p_2)(1-p_1 p_2 p_3) e^{t_2}}{(1-p_2 e^{t_2})(1-p_1 p_2 p_3 e^{t_2})}$$

which is the MGF of $Y \sim WG\left(\frac{lnp_1 p_3}{lnp_2}, p_2\right)$.

4. New Bivariate Weighted Geometric Distribution

Let V_1, V_2 and V_3 are independent random variables, where V_1 and V_2 are geometric distribution with distribution function,

$$F_{V_i}(v) = 1 - {p_i}^{v+1}, i = 1,2, v = 0,1,2, \dots$$

and V_3 be a generalized discrete exponential distribution with parameters p_3 and α having distribution function,

$$F_{V_3}(v) = (1 - {p_i}^{v+1})^{\alpha}, v = 0,1,2, \dots, \alpha > 0.$$

Let $(W, Z) \stackrel{d}{=} \big((V_1, V_2) | V_3 < min(V_1, V_2)\big)$. Then (W, Z) is a new bivariate weighted geometric distribution with parameters p_1, p_2, p_3 and α, which is denoted by $NBWGD(p_1, p_2, p_3, \alpha)$.

a) The Joint Probability Mass Function

Let $(W,Z)\sim NBWGD(p_1,p_2,p_3,\alpha)$, then joint probability mass function of (W,Z) is

$$f_{NBWGD}(w,z)=\frac{({p_1}^{w}-{p_1}^{w+1})({p_2}^{z}-{p_2}^{z+1})(1-{p_3}^{r})^{\alpha}}{1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}\frac{1-p_1p_2}{1-p_1p_2{p_3}^{t}}},$$

where $r=min(w,z)$.

Proof:

We have

$$f_{NBWGD}(w,z)=\frac{P(V_1=w)P(V_2=z)P(V_3<min(V_1,V_2)|V_1=w,V_2=z)}{\sum_{w=0}^{\infty}\sum_{z=0}^{\infty}P(V_1=w)P(V_2=z)P(V_3<min(V_1,V_2)|V_1=w,V_2=z)}$$
$$=\frac{({p_1}^{w}-{p_1}^{w+1})({p_2}^{z}-{p_2}^{z+1})\left(1-{p_3}^{min(w,z)}\right)^{\alpha}}{\sum_{w=0}^{\infty}\sum_{z=0}^{\infty}({p_1}^{w}-{p_1}^{w+1})({p_2}^{z}-{p_2}^{z+1})(1-{p_3}^{min(w,z)})^{\alpha}}.$$

Using the generalized binomial theorem, we have

$$\left(1-{p_3}^{min(w,z)}\right)^{\alpha}=1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}{p_3}^{t\,min(w,z)}.$$

Thus,

$$f_{NBWGD}(w,z)=\frac{({p_1}^{w}-{p_1}^{w+1})({p_2}^{z}-{p_2}^{z+1})1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}{p_3}^{tr}}{\sum_{w=0}^{\infty}\sum_{z=0}^{\infty}({p_1}^{w}-{p_1}^{w+1})({p_2}^{z}-{p_2}^{z+1})\left(1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}{p_3}^{t}\right)}.$$

where $r=min(w,z)$ and

$$K^{-1}=({p_1}^{w}-{p_1}^{w+1})({p_2}^{z}-{p_2}^{z+1})\left(1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}{p_3}^{t\,min(w,z)}\right)$$
$$=(1-p_1)(1-p_2)\left[\sum_{w=0}^{\infty}\sum_{z=w+1}^{\infty}{p_1}^{w}{p_2}^{z}\left(1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots\left(\alpha-(t-1)\right)}{t}(-1)^{t-1}{p_3}^{wt}\right)\right.$$
$$+\sum_{z=0}^{\infty}\sum_{w=z+1}^{\infty}{p_1}^{w}{p_2}^{z}\left(1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots\left(\alpha-(t-1)\right)}{t}(-1)^{t-1}{p_3}^{zt}\right)$$
$$\left.+\sum_{w=0}^{\infty}(p_1p_2)^{w}\left(1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\ldots\left(\alpha-(t-1)\right)}{t}(-1)^{t-1}{p_3}^{wt}\right)\right]$$

$$= (1-p_1)(1-p_2)\Bigg[\frac{p_2}{1-p_2}\Bigg((1-p_1p_2)^{-1} - \sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}(1-p_1p_2{p_3}^t)^{-1}\Bigg) + \frac{p_1}{1-p_1}\Bigg((1-p_1p_2)^{-1} - \sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}(1-p_1p_2{p_3}^t)^{-1}\Bigg) + (1-p_1p_2)^{-1} - \sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}(1-p_1p_2{p_3}^t)^{-1}\Bigg]$$

$$=(1-p_1)(1-p_2)\left[\left(1+\frac{p_1}{1-p_1}+\frac{p_2}{1-p_2}\right)\left((1-p_1p_2)^{-1} - \sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}(1-p_1p_2{p_3}^t)^{-1}\right)\right]$$

$$= 1 - \sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}\frac{1-p_1p_2}{1-p_1p_2{p_3}^t}.$$

The joint probability mass function becomes

$$f_{NBWGD}(w,z) = \frac{({p_1}^w - {p_1}^{w+1})({p_2}^z - {p_2}^{z+1})\left(1-{p_3}^{min(w,z)}\right)^{\alpha}}{1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}\frac{1-p_1p_2}{1-p_1p_2{p_3}^t}}$$

or $f_{NBWGD}(w,z) = \frac{({p_1}^w - {p_1}^{w+1})({p_2}^z - {p_2}^{z+1})(1-{p_3}^r)^{\alpha}}{1-\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}\frac{1-p_1p_2}{1-p_1p_2{p_3}^t}}$,

where $r = min(w,z)$.

b) The Joint Distribution Function

The distribution function of new bivariate discrete weighted geometric distribution with parameters p_1, p_2, p_3 and α is

$$F_{NBWGD}(w,z) = \frac{1}{\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^t)(1-p_1p_2{p_3}^t)^{-1}}$$
$$\left[\sum_{t=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^t) * \frac{1-(p_1p_2{p_3}^t)^{r+1}}{1-p_1p_2{p_3}^t} - \frac{1-(p_1{p_3}^t)^{r+1}}{1-p_1{p_3}^t}{p_2}^z - \frac{1-(p_2{p_3}^t)^{r+1}}{1-p_2{p_3}^t}{p_1}^w + \frac{1-{p_3}^{t(r+1)}}{1-p_1{p_3}^t}{p_1}^w{p_2}^z\right]$$

Proof:

By definition

$$\begin{aligned} F_{NBWGD}(w,z) &= P(W \le w, Z \le z) \\ &= P\big(V_1 \le w, V_2 \le z \big| V_3 < min(V_1,V_2)\big) \\ &= \frac{P(V_3 < V_1 \le w, V_3 < V_2 \le z)}{P(V_3 < min(V_1,V_2))}, \end{aligned}$$

where

$$P(V_3 < V_1 \leq w, V_3 < V_2 \leq z)$$
$$= \sum_{u=0}^{r} ({p_1}^{u+1} - {p_1}^{w+1})({p_2}^{u+1} - {p_2}^{z+1})(\,(1-{p_3}^{u+1})^{\alpha} - (1-{p_3}^{u})^{\alpha})$$
$$= \sum_{u=0}^{r} ({p_1}^{u+1} - {p_1}^{w+1})({p_2}^{u+1} - {p_2}^{z+1}) \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}{p_3}^{tu}(1-{p_3}^{t})$$
$$= \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t}) \sum_{u=0}^{r} ({p_1}^{u+1} - {p_1}^{w+1})({p_2}^{u+1} - {p_2}^{z+1}){p_3}^{tu}$$
$$= p_1p_2 \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t}) * \left[\sum_{u=0}^{r} ((p_1p_2{p_3}^{t})^{u} - (p_1{p_3}^{t})^{u}{p_2}^{z} - {p_1}^{w}(p_2{p_3}^{t})^{u} + {p_1}^{w}{p_2}^{z}{p_3}^{ut})\right]$$

$$= p_1p_2 \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t}) * \left[\frac{1-(p_1p_2{p_3}^{t})^{r+1}}{1-p_1p_2{p_3}^{t}} - \frac{1-(p_1{p_3}^{t})^{r+1}}{1-p_1{p_3}^{t}}{p_2}^{z} - \frac{1-(p_2{p_3}^{t})^{r+1}}{1-p_2{p_3}^{t}}{p_1}^{w} + \frac{1-{p_3}^{t(r+1)}}{1-{p_3}^{t}}{p_1}^{w}{p_2}^{z}\right]$$

and

$$P(V_3 < min(V_1, V_2)) = \sum_{u=0}^{\infty} (p_1p_2)^{u+1} \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}{p_3}^{ut}(1-{p_3}^{t})$$
$$= p_1p_2 \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t}) \sum_{u=0}^{\infty} (p_1p_2{p_3}^{t})^{u}$$
$$= p_1p_2 \sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t})\,(1-p_1p_2{p_3}^{t})^{-1}.$$

Thus, the joint distribution function of the $NBWGD(p_1, p_2, p_3, \alpha)$ becomes

$$F_{NBWGD}(w,z) = \frac{1}{\sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t})(1-p_1p_2{p_3}^{t})^{-1}}$$
$$\left[\sum_{t=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(-1)^{t-1}(1-{p_3}^{t}) * \frac{1-(p_1p_2{p_3}^{t})^{r+1}}{1-p_1p_2{p_3}^{t}} - \frac{1-(p_1{p_3}^{t})^{r+1}}{1-p_1{p_3}^{t}}{p_2}^{z} - \frac{1-(p_2{p_3}^{t})^{r+1}}{1-p_2{p_3}^{t}}{p_1}^{w} + \frac{1-{p_3}^{t(r+1)}}{1-p_1{p_3}^{t}}{p_1}^{w}{p_2}^{z}\right].$$

c) Moment Generating Function

If $(W,Z) \sim NBWGD(p_1, p_2, p_3, \alpha)$ then the moment generating function of (W,Z) is given by

$$M_{NBWGD}(t_1,t_2) = K\frac{(1-p_1)(1-p_2)}{(1-p_1e^{t_1})(1-p_2e^{t_2})}\left[1 - \sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))(1-p_1p_2e^{t_1+t_2})}{t(1-p_1p_2{p_3}^te^{t_1+t_2})}\right]$$

where

$$K^{-1} = 1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))(1-p_1p_2)}{t(1-p_1p_2{p_3}^t)}.$$

Proof:

Let $(W,Z)\sim NBWGD(p_1,p_2,p_3,\alpha)$ then the moment generating function of (W,Z) can be derived as

$$M_{NBWGD}(t_1,t_2) = \sum_{w=0}^{\infty}\sum_{z=0}^{\infty}e^{t_1w+t_2z}f_{NBWGD}(w,z)$$

$$= K^{-1}\sum_{w=0}^{\infty}\sum_{z=0}^{\infty}\left(e^{t_1w+t_2z}({p_1}^w-{p_1}^{w+1})({p_2}^z-{p_2}^{z+1})\right)$$

$$*\left[1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1)){p_3}^{tmin(w,z)}}{t}\right]$$

$$= K^{-1}(1-p_1)(1-p_2)\left[\sum_{w=0}^{\infty}\sum_{z=w+1}^{\infty}\left(e^{t_1w+t_2z}{p_1}^w{p_2}^z\left(1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1)){p_3}^{tw}}{t}\right)\right)\right.$$

$$+\sum_{z=0}^{\infty}\sum_{w=z+1}^{\infty}\left(e^{t_1w+t_2z}{p_1}^w{p_2}^z\left(1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots\big(\alpha-(t-1)\big){p_3}^{tz}}{t}\right)\right)$$

$$\left.+\sum_{w=0}^{\infty}\left(e^{t_1w+t_2z}(p_1p_2)^w\left(1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots\big(\alpha-(t-1)\big){p_3}^{tw}}{t}\right)\right)\right]$$

On simplification we get,

$$M_{NBWGD}(t_1,t_2) = K\frac{(1-p_1)(1-p_2)}{(1-p_1e^{t_1})(1-p_2e^{t_2})}\left[1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))(1-p_1p_2e^{t_1+t_2})}{t(1-p_1p_2{p_3}^te^{t_1+t_2})}\right]$$

where

$$K^{-1} = 1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))(1-p_1p_2)}{t(1-p_1p_2{p_3}^t)}.$$

d) Properties of the New Bivariate Weighted Geometric Distribution

Discrete Analogue of Bivariate Weighted Generalized Exponential Distribution

If (X, Y) is a bivariate weighted generalized exponential distribution proposed by Ghosh and Alzaatreh (2016) with joint probability density function

$$f(x,y) = \theta^{-2}\left(\frac{1}{\alpha_0+\alpha_1+\alpha_2}\left(\frac{1}{\alpha_0+\alpha_1}+\frac{1}{\alpha_0+\alpha_2}\right)\right)^{-1} e^{-\left(\frac{x}{\theta}+\frac{y}{\theta}\right)}$$
$$*\left(1-e^{-\left(\frac{x}{\theta}\right)}\right)^{\alpha_1-1}\left(1-e^{-\left(\frac{y}{\theta}\right)}\right)^{\alpha_2-1}\left(1-e^{-\left(\frac{min(x,y)}{\theta}\right)}\right)^{\alpha_0},$$

where x > 0, y > 0 and $\alpha_i >\ 0$, i=0,1,2 and θ> 0 are the parameters of the distribution. If, in particular the parameters of the bivariate weighted generalized exponential distribution, α_i = 1; i = 1; 2 and W = [X] and Z = [Y] and then (W; Z) follows new bivariate weighted geometric distribution with parameters $p_1 = p_2 = p_3 = e^{\theta}$ and α_0.

TP_2 Property

For every pair (w; z) from $N_0 x N_0$, the joint probability mass function of new bivariate weighted geometric distribution, $f_{\text{NBWGD}}(w,z)$ satisfies

$$\frac{f_{\text{NBWGD}}(w_{11},z_{21})f_{\text{NBWGD}}(w_{12},z_{22})}{f_{\text{NBWGD}}(w_{12},z_{21})f_{\text{NBWGD}}(w_{11},z_{22})} \geq 1.$$

According to Ali *et al.* (2019), we can say that the joint probability mass function of new bivariate weighted geometric distribution holds a total positivity of order two property.

Marginal Density

If (W, Z) is bivariate weighted geometric distribution, then the probability mass function and distribution function of Z can be derived as,

$$f_Z(z) = K^{-1}(1-p_2)p_2{}^z\left(1+\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}(1-p_1(1+(p_1p_3{}^t)^z(1-p_3{}^t)))\right)$$

and

$$F_Z(z) = K^{-1}\left(1-p_2{}^{z+1}+\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))}{t}\frac{(1-p_1p_2p_3)(1-p_1)(1-p_2{}^{z+1})-p_1(1-p_2)(1-p_3{}^t)(1-p_1p_2p_3{}^t)^{z+1}}{1-p_1p_2p_3}\right)$$

where

$$K^{-1} = 1-\sum_{t=1}^{\infty}(-1)^{t-1}\frac{\alpha(\alpha-1)\ldots(\alpha-(t-1))(1-p_1p_2)}{t(1-p_1p_2p_3{}^t)}.$$

e) Application of the Distribution

Application in Risk Model

Consider a system with two components 1 and 2, which fail due to two independent causes A and B, and let C be an extraneous strength independent of the causes A and B. Let the system only fail if the causes overcome the strength C. Assume that the failure time of the components is discrete, say X for component 1 and Y for component 2. If U_1, U_2 and U_3 be the lifetime of the components due to cause A and B and strength C and U_1 and U_2 are geometric distributions with parameters p_1 and p_2 respectively and U_3 is a generalized discrete exponential distribution with parameters p_3 and α, then

$(X,Y)\overset{d}{=}(U_1,U_2)|U_3 < min(U_1,U_2)$ follows a bivariate weighted geometric distribution with parameters p_1, p_2, p_3 and α.

Application in Shock Model

Suppose a system contains two components 0 and 1 and assume that the system receives shocks from two different sources, say source A and source B. The system receives shocks randomly and independently of other shocks discretely from the sources. Let there be a hindrance from source C such that the system will receive shock only if the strength of the shocks from sources A and B overcome the stress from source C. Let U_1 and U_2 denote the discrete times at which shocks appear from source A and source B, respectively and the system will fail when it receives first shock. If U_1 follows a geometric distribution with parameter p_1, U_2 is a geometric with parameter p_2 and U_3 is a generalized discrete exponential with parameters p_3 and α, then $(X,Y)\overset{d}{=}(U_1,U_2)|U_3 < min(U_1,U_2)$ follows a bivariate weighted geometric distribution with parameters p_1, p_2, p_3 and α.

5. Simulation Study

The joint cumulative distribution function $F_{X,Y}(x,y)$ is used to describe a bivariate random variable (X, Y). Consider statistically independent standard uniform variables U_1 and U_2. Let (u_1,u_2) is a random sample from $U = (U_1,U_2)$, such that

$$u_1 = F_Y(y)$$
$$u_2 = F_{X/Y=y}(x)$$

where $F_Y(y)$ and $F_{X/Y=y}(x)$ are distribution functions of Y and conditional distribution function of X given Y = y, respectively. By inverting these two equations, we get the values of the bivariate random variable. Since $Y \sim WG\left(\frac{ln(p_1p_3)}{ln(p_2)}, p_2\right)$,

$$F_Y(y) = \sum_{t=0}^{y} \frac{(1-p_2)(1-p_1p_2p_3){p_2}^t(1-(p_1p_3)^{t+1})}{1-p_1p_3}$$
$$= \frac{(1-p_2)(1-p_1p_2p_3)}{1-p_1p_3}\left(\frac{1-{p_2}^{y+1}}{1-p_2} - p_1p_3\frac{1-(p_1p_2p_3)^{y+1}}{1-p_1p_2p_3}\right).$$
$$F_{X/Y=y}(x) = P(X \le x/Y = y) = \frac{1}{f_Y(y)}\sum_{t\le x} f(t,y),$$

where $f_Y(y)$ is the probability mass function of the marginal distribution of Y. Also, using the following two equations

$$f(x,y) = S(x,y) - S(x,y+1) - S(x+1,y) + S(x+1,y+1)$$

and

$$S(x,y) = F(x,y) - F_X(x) - F_Y(y) + 1$$

where f (x, y) is the joint probability mass function of (X,Y), $S(x,y)$ is the joint survival function of (X, Y), F(x, y) is the joint distribution function of (X, Y), $F_X(x)$ is the marginal distribution function of X and $F_Y(y)$ is the marginal distribution function of Y, we get

$$\sum_{t\le x} f(t,y) = \sum_{t\le x}[S(t,y) - S(t,y+1) - S(t+1,y) + S(t+1,y+1)]$$
$$= S(0,y) - S(0,y+1) - S(x+1,y) + S(x+1,y+1)$$
$$= F(0,y) - F(0,y+1) - F(x+1,y) + F(x+1,y+1)$$

$=$
$(p_1p_2p_3)^{min(x+1,y)+1}+p_2{}^y(1-p_1p_2p_3)\frac{1-(p_1p_3)^{min(x+1,y)+1}}{1-p_1p_3}+$
$p_1{}^{x+1}(1-p_1p_2p_3)\frac{1-(p_2p_3)^{min(x+1,y)+1}}{1-p_2p_3}-p_1{}^{x+1}p_2{}^y(1-p_1p_2p_3)\frac{1-p_3{}^{min(x+1,y)+1}}{1-p_3}-$
$(p_1p_2p_3)^{min(x+1,y+1)+1}-p_2{}^{y+1}(1-p_1p_2p_3)\frac{1-(p_1p_3)^{min(x+1,y+1)+1}}{1-p_1p_3}-p_1{}^{x+1}(1-$
$p_1p_2p_3)\frac{1-(p_2p_3)^{min(x+1,y+1)+1}}{1-p_2p_3}+p_1{}^{x+1}p_2{}^{y+1}(1-p_1p_2p_3)\frac{1-p_3{}^{min(x+1,y+1)+1}}{1-p_3}.$

Thus,

$$F_{X/Y=y}(x)=\frac{1-p_1p_3}{(1-p_2)(1-p_1p_2p_3)p_2{}^t(1-(p_1p_3)^{t+1})}$$

$*\Big[(p_1p_2p_3)^{min(x+1,y)+1}+p_2{}^y(1-p_1p_2p_3)\frac{1-(p_1p_3)^{min(x+1,y)+1}}{1-p_1p_3}+$
$p_1{}^{x+1}(1-p_1p_2p_3)\frac{1-(p_2p_3)^{min(x+1,y)+1}}{1-p_2p_3}-p_1{}^{x+1}p_2{}^y(1-p_1p_2p_3)\frac{1-p_3{}^{min(x+1,y)+1}}{1-p_3}-$
$(p_1p_2p_3)^{min(x+1,y+1)+1}-p_2{}^{y+1}(1-p_1p_2p_3)\frac{1-(p_1p_3)^{min(x+1,y+1)+1}}{1-p_1p_3}-p_1{}^{x+1}(1-$
$p_1p_2p_3)\frac{1-(p_2p_3)^{min(x+1,y+1)+1}}{1-p_2p_3}+p_1{}^{x+1}p_2{}^{y+1}(1-p_1p_2p_3)\frac{1-p_3{}^{min(x+1,y+1)+1}}{1-p_3}\Big].$

Based on Lai *et al.* (2013), the correlated variables (X; Y) can be obtained as

$$y=F_Y{}^{-1}(\ u_1)$$
$$x=F_{X/Y=y}{}^{-1}(u_2).$$

So, we can study the performance of the MLEs of the bivariate weighted geometric distribution with parameters, p_1, p_2 and p_3 for various sample sizes and for selected parameter values with 250 iterations using Monte Carlo simulation. To estimate the parameters, we have to solve three nonlinear equations. We use R program for estimation and simulating sample values. Also, bias and mean square error (MSE) are obtained for each case.

Similarly, we can carry out the simulation study of the new bivariate weighted geometric distribution with parameters p_1, p_2, p_3 and α. Since the simulation study is carried out using R program, the main objective of this section is to define the normalizing constant. The normalizing constant is an infinite sum of functions of p_1, p_2, p_3 and α. Since the parameters $p_1, p_2,$ and p_3 take values from (0,1), the sum $\sum_{w=0}^{N}\sum_{z=0}^{N}(p_1{}^w-p_1{}^{w+1})(p_2{}^z-p_2{}^{z+1})\left(1-p_3{}^{min(w,z)}\right)^{\alpha}$ tends to a finite value for a large N for given values of p_1, p_2, p_3 and α. So, we use sum $\sum_{w=0}^{N}\sum_{z=0}^{N}(p_1{}^w-p_1{}^{w+1})(p_2{}^z-p_2{}^{z+1})(1-p_3{}^{min(w,z)})^{\alpha}$ instead of sum $\sum_{w=0}^{\infty}\sum_{z=0}^{\infty}(p_1{}^w-p_1{}^{w+1})(p_2{}^z-p_2{}^{z+1})\left(1-p_3{}^{min(w,z)}\right)^{\alpha}$ for large N (say, N=10^5).

Table 1: Simulation study for parameter set ($p_1=0.49, p_2=0.67, p_3=0.55$).

Sample Size		$p_1=0.49$	$p_2=0.67$	$p_3=0.55$
n=80	Estimate	0.33997	0.66102	0.39834
	Bias	-0.15003	-0.00898	-0.15166
	MSE	0.09417	0.00158	0.09852

n=150	Estimate Bias MSE	0.34938 -0.14062 0.08452	0.66161 -0.000839 0.00115	0.41099 -0.13901 0.08833
n=180	Estimate Bias MSE	0.3516 -0.1384 0.07461	0.66265 -0.00735 0.00087	0.41558 -0.13442 0.07892
n=200	Estimate Bias MSE	0.35782 -0.13218 0.07068	0.6627 -0.00073 0.00086	0.42388 -0.12612 0.0736
n=240	Estimate Bias MSE	0.3622 -0.1278 0.0634	0.66388 -0.00612 0.00069	0.43362 -0.11638 0.06324
n=280	Estimate Bias MSE	0.37028 -0.11972 0.00581	0.66428 -0.00572 0.00063	0.44234 -0.10766 0.05558

Table 2: Simulation study of new bivariate weighted geometric distribution for parameter set ($p_1 = 0.965, p_2 = 0.15, p_3 = 0.8, \alpha = 1.5$).

Sample size		$p_1 = 0.965$	$p_2 = 0.15$	$p_3 = 0.8$	$\alpha = 1.5$
n=40	Estimate Bias MSE	0.922 0.043 0.0018	0.245 -0.095 0.009	0.548 0.252 0.063	1.983 -0.483 0.233
n=80	Estimate Bias MSE	0.925 0.04 0.0016	0.199 -0.049 0.0022	0.656 0.144 0.021	1.912 -0.412 0.169
n=120	Estimate Bias MSE	0.932 0.033 0.0011	0.195 -0.045 0.002	0.664 0.136 0.018	1.784 -0.284 0.081
n=160	Estimate Bias MSE	0.938 0.027 0.0007	0.188 -0.038 0.0014	0.694 0.106 0.011	1.716 -0.216 0.047
n=200	Estimate Bias MSE	0.942 0.023 0.0005	0.176 -0.026 0.0007	0.710 0.09 0.0082	1.692 -0.192 0.037

6. Estimation

To estimate unknown parameters p_1, p_2, p_3 and α of new bivariate weighted geometric distribution using maximum likelihood estimation, consider a sample of size n, $\{(w_{11}, w_{21}), (w_{12}, w_{22}), \dots, (w_{1n}, w_{2n})\}$ from new bivariate weighted geometric distribution.

$$I_1 = \{i; w_{1i} > w_{2i}\}, I_2 = \{i; w_{1i} < w_{2i}\}\ and\ I_0 = \{i; w_{1i} = w_{2i} = w_i\}$$

and n_j is the number of elements in the set $I_j, j = 0,1,2. n = n_0 + n_1 + n_2$.

The likelihood function $l(w; p_1, p_2, p_3, \alpha)$ is given by

$$l(x; p_1, p_2, p_3, \alpha) = \prod_{j=1}^{n_1} f_1(w_{1j}, w_{2j}) \prod_{j=1}^{n_2} f_2(w_{1j}, w_{2j}) \prod_{j=1}^{n_0} f_0(w_{1j})$$

Thus, the log likelihood function $L(x; p_1, p_2, p_3, \alpha)$,

$$L(w; p_1, p_2, p_3, \alpha) = \sum_{j \in I_0 \cup I_1} \left[\sum_{k=1}^{2} log(p_k{}^{w_{kj}} - p_k{}^{w_{kj}+1}) + \alpha log(1 - p_3{}^{w_{1j}}) \right]$$

$$+ \sum_{j=1}^{n_2} \left[\sum_{k=1}^{2} log(p_k{}^{w_{kj}} - p_k{}^{w_{kj}+1}) + \alpha log(1 - p_3{}^{w_{2j}}) \right]$$

$$+ nlog \left[\sum_{w_1}^{\infty} \sum_{w_2}^{\infty} (p_1{}^{w_1} - p_1{}^{w_1+1})(p_2{}^{w_2} - p_2{}^{w_2+1})(1 - p_3{}^{min(w_1+w_2)})^{\alpha} \right]$$

By solving the normal equations

$$\frac{\partial L(w; p_1, p_2, p_3, \alpha)}{\partial p_1} = 0$$

$$\frac{\partial L(w; p_1, p_2, p_3, \alpha)}{\partial p_2} = 0$$

$$\frac{\partial L(w; p_1, p_2, p_3, \alpha)}{\partial p_3} = 0$$

and

$$\frac{\partial L(w; p_1, p_2, p_3, \alpha)}{\partial \alpha} = 0,$$

we get the maximum likelihood estimate of parameters p_1, p_2, p_3 and α.

7. Real Data Analysis

Here, we use two real life data sets to show the applicability of the proposed distributions. Further, the new bivariate weighted geometric distribution and its special cases are compared with bivariate geometric distribution and bivariate weighted geometric distribution proposed by Najarzadegan *et al*. (2018). In order to compare the distributions we consider the log-likelihood, AIC, BIC values. Since the maximum likelihood estimate is not unique, we take bootstrap samples from the new bivariate weighted geometric distribution function with preassigned parameters. Then, we find the maximum likelihood estimate of the parameters of the simulated samples. Using these estimates of parameters as initial values, the estimates of the parameters for the corresponding data set are obtained. Average of the estimates of the corresponding parameters is taken as the maximum likelihood estimator.

Data 1

Here we first consider the scores taken from a video recorded during the summer of 1995 relayed by NBC sports TV, IX World Cup diving competition, Atlanta, Georgia (Najarzadegan *et al.* (2018)).

Table 3: Analysis using data set 1.

Distribution	p_1	p_2	p_3	p_4	α	-log	AIC	BIC
BGE	0.9616	0.9854	0.9401	-	-	117.64	241.28	244.11
WBGE	0.9385	0.9764	0.9093	0.999	-	106.17	220.34	224.11
BWGD	0.9331	0.935	0.999	-	-	105.97	217.95	220.79
NBWGD	0.93309	0.9349	0.9990	-	1.093	102.79	213.86	217.36

Data 2

The second data set is score taken from a video recorded during the summer of 1995 relayed by NBC sports TV, IX world cup diving competition, Atlanta, Georgia (Dhar (2003)) From these two tables, we can say that the new bivariate weighted geometric distribution and its special case are better fit to this data.

Table 4: Analysis using data set 2.

Distribution	p_1	p_2	p_3	p_4	α	-log	AIC	BIC
BGE	0.984	0.982	0.997	-	-	247.46	500.92	504.58
WBGE	0.976	0.973	0.996	0.99	-	232.33	472.64	477.51
BWGD	0.982	0.981	0.999	-	-	171.47	348.94	352.6
NBWGD	0.982	0.981	0.999	-	1.45	136.198	280.3	285.27

8. Conclusion

Here, we introduced a new bivariate weighted geometric distribution and a special case of the distribution with three parameters and some of its important properties, such as cumulative distribution function, joint probability mass function, marginal distributions, moment generating function etc. The maximum likelihood estimates of the parameters of the distribution are obtained. Estimation of parameters is illustrated using two real life data sets. The simulation study is also carried out.

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