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Critical Connectivity Parameter for One Dimensional Random Geometric Graph from Power Law Densities

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ABSTRACT

A random geometric graph is defined as follows: for a set of random points in some set $A \subseteq \mathbb{R}^d$, fix $r > 0$ and connect a pair of points X, Y, provided that $\| X - Y \| < r$. The graph generated with the set of points as the vertices and the random set of edges thus constructed, is called random geometric graph. Previous studies have been conducted under the assumption that the random set of points originate as an i.i.d. sample from a density function that is bounded away from zero on A. We restrict our attention to densities on $[0,1]$ having a zero at the origin. Under this assumption, we study the asymptotic behaviour of connectivity distance of the random geometric graph. Our results show that the connectivity distance behaves differently as compared to the case of densities that are bounded away from zero.

1. Introduction and Statement of Results

Phenomenon of random geometric graphs arise quite naturally. For example, there is a network of communication stations distributed across the country and people connect to the stations according to the proximity of a station. The spread of forest fire depends on the proximity of the neighboring trees. A neural network is viewed as computation units with connections between nearby layers. The constellations of stars are grouped according to their positions in the sky. Clustering principles are based on the proximity of two observations. It is natural for a statistician to model such situations and make numerical measurements so as to study different aspects of the network. These models would be based on the geometry of underlying spaces and the random geometric graph becomes an integral part of the model.

Gilbert (1961) first introduced the random geometric graph to model the communications between radio stations. Gilbert's original model was defined on

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the unbounded space as follows: pick points in $\mathbb{R}^2$ according to a Poisson Point Process of intensity one and join two points if their distance is less than some parameter $r > 0$. The Gilbert model has given rise to the percolation theory in the continuum which has been intensively studied in the last few decades.
The most closely related model of the Gilbert's model is where n nodes are independently and identically distributed on the space. A random geometric graph is formally defined as follows: let $\mathcal{P}_n$ be a set of random points in some set $A \subseteq \mathbb{R}^d$ for $d \geq 1$. Fix $r > 0$. For a pair of points $X, Y \in \mathcal{P}_n$, the points are connected by an un-directed edge if $\parallel X - Y \parallel < r$. This results in a random graph with the vertex set $\mathcal{P}_n$ and the random edge set generated by the mechanism given above. We denote the above graph by $G(\mathcal{P}_n, r)$.

The random geometric graph has been studied extensively in the last decade. A large body of literature has been devoted to studying the properties of low-dimensional random geometric graphs (see Dall and Christensen (2002), Penrose (2003), Bollobás (2001)). Random geometric graphs have found wide applications in a very large span of fields. One can mention wireless networks (Haenggi *et al.* (2009), Mao and Anderson (2012)), gossip algorithms (Wang and Lin (2018)), consensus building (Estrada and Sheerin (2016)), spread of a virus (Preciado and Jadbabaie (2009)), protein-protein interactions (Higham *et al.* (2008)), citation networks (Xie *et al.* (2016)), robotics (Solovey *et al.* (2018)) etc. The ubiquity of this random graph model to faithfully represent real world networks has motivated a great interest in its theoretical study.

The most commonly occurring random geometric graphs arise from the independent and identically distributed points in the subset of $\mathbb{R}^d$. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common density f on $\mathbb{R}$. In this article, we consider the common density f of the sequence belonging to a particular class, which we will introduce shortly. From any realizations of the sequence $\{X_n : n \geq 1\}$, we construct the random geometric graph. For every $n \geq 1$, we consider the first n realizations, i.e., let us set

$$\mathcal{P}_n = \{X_1, X_2, \ldots, X_n\}$$

With $\mathcal{P}_n$, defined above, and $r > 0$, we construct the graph $G(\mathcal{P}_n, r)$ as described above, i.e., connect two vertices X_i and X_j of $\mathcal{P}_n$ by an edge if and only $|X_i - X_j| < r$.

When the sample observations are taken from a distribution having a density which is bounded below, a lot of results are known. Walters (2011) has provided a survey of the properties of the random geometric graph including connectivity, giant component, coverage or chromatic number. Two limiting regimes in the random geometric graph are of special interest. One of these is the thermodynamic limit in which the expected degree of a typical vertex tends to a constant. In this regime, in the limiting version, one observes n points in a region of volume proportional to n, with their inter-point distances becoming roughly constant. If the limiting inter point distance in the random geometric graph exceeds a certain critical value then there is likely to be existence of giant component.

The second limiting regime is the connectivity regime, which is the special case of the dense limit regime. Here the typical vertex degree grows logarithmically in n. Clearly, a necessary condition for connectivity is that there will be no isolated points, and this turns out to be sufficient with high probability as $n \to \infty$. We refer the readers to Penrose (2003) for more details.

We want to study the connectivity property of the random geometric graph. A graph is called connected if it has only one component. For un-directed graphs, it is easy to see that the graph is connected if from every vertex, we can reach any other vertex using a finite set of edges. In this case, we clearly see that as the value of the parameter r increases, the more pairs of edges get connected. Hence the connectivity is an increasing property of r. Therefore, it makes sense to consider the minimum value of r above which the graph is connected. Let $r_c(n; f_\alpha)$ be the critical connectivity parameter defined by

$$r_c(n; f) = \inf\{r > 0: G(\mathcal{P}_n, r) \text{ is connected }\} \tag{1.1}$$

Since the set of vertices $\mathcal{P}_n$ are coming from a continuous density, no two points are same. Hence, for every $n \geq 1$, we see that

$$\min\{|X_i - X_j|: 1 \leq i, j \leq n\} > 0$$

So, it is the case that $r_c(n; f) \geq \min\{|X_i - X_j|: 1 \leq i, j \leq n\} > 0$. Also, when $r = 1$, all pairs are trivially connected. Therefore, we have $r_c(n; f) \leq 1$. In other words, the critical value $r_c(n; f_\alpha)$ is non-trivial. It is intuitively obvious that as the number of vertices increase the critical connectivity parameter will decrease. Our aim is to show how the critical value behaves as n approaches infinity.

Previous works including Penrose (2003) have been devoted to the case where the density is bounded below. In this article, we consider the class of power law densities on [0,1] which admits a zero. It is natural to consider the family of densities given by $\beta x^{\beta-1}$ for $\beta > 0$. However, for $0 < \beta \leq 1$, this density is bounded away from 0 and hence is already covered by the results of Penrose (2003). Therefore, we need to consider the case when $\beta > 1$. Thus, taking $\alpha = \beta - 1 > 0$, we have the density function f_α given by

$$f_\alpha(x) = (1+\alpha)x^\alpha \text{ for } x \in [0,1] \qquad (1.2)$$

Here $\alpha > 0$ may be taken as the parameter.

The following simulations show that the behaviour of the critical connectivity parameter behaves very differently than the case usually considered, though for different values of α, the simulation shows remarkable similarity in shape and type of distribution.

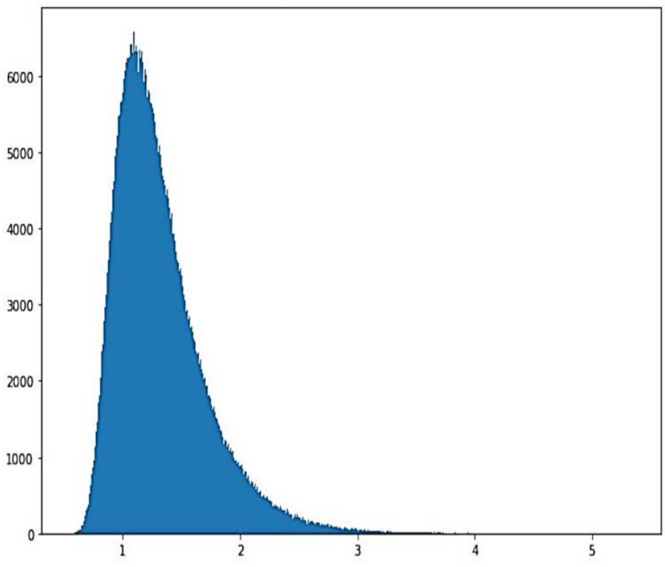

Fig. 1: Critical Connectivity distance with $\alpha = 0.5$ and $N = 10000$.

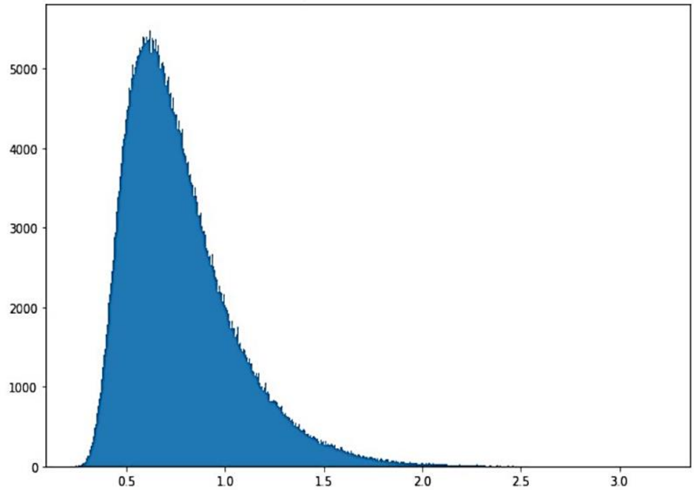

Fig. 2: Critical Connectivity distance with $\alpha = 1$ and $N = 10000$.

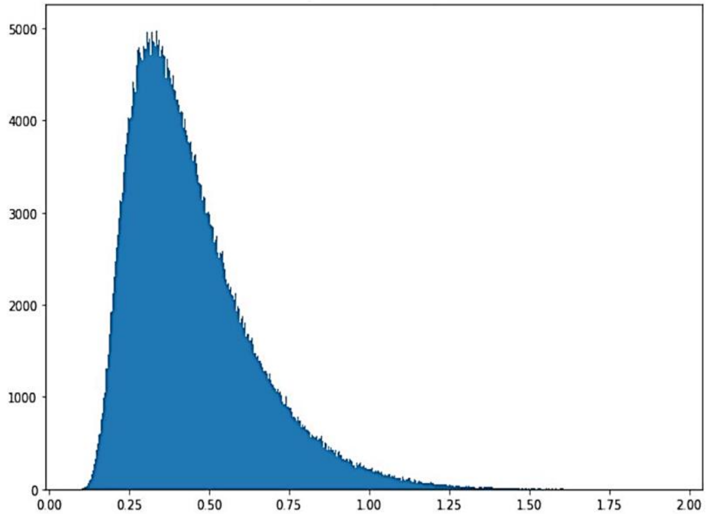

Fig. 3: Critical Connectivity distance with $\alpha = 2$ and $N = 10000$.

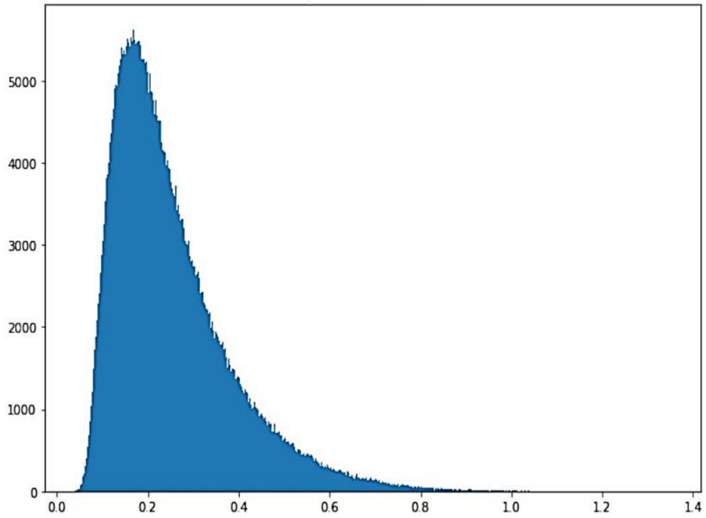

Fig. 4: Critical Connectivity distance with $\alpha = 4$ and $N = 10000$.

Before we state our result, we need to introduce the limiting random variables. Let $E_1, E_2, \ldots$ be a sequence of independent and identically distributed random random variables with each distributed as an exponential distribution with parameter 1. Let

us define the partial sums, i.e., let

$$S_k = \sum_{i=1}^{k} E_i \tag{1.3}$$

For any $\beta > 1$, let Y_β be defined by

$$Y_\beta = \sup\left\{S_{k+1}^{1/\beta} - S_k^{1/\beta} : k \geq 1\right\} \tag{1.4}$$

Note that our results will involve Y_β. Therefore, we need to show that Y_β is a proper random variable. We relegate to the next section (see Lemma 1).

Theorem 1: Let $\{X_i : 1 \leq i \leq n\}$ be i.i.d. samples from a distribution having density f_α for $\alpha > 0$ and let $r_c(n; f_\alpha)$ denote the critical parameter for the random geometric graph to become connected. Then,

$$n^{1/(1+\alpha)} r_c(n; f_\alpha) \Rightarrow Y_{1+\alpha} \tag{1.5}$$

where $Y_{1+\alpha}$ is as defined above.

At this point, we have no further information on the limiting random variable Y_β. However simulation results show (see Fig. 5, 6, 7 and 8) that the limiting random variable should be continuous, unimodal with sharply decaying tail probabilities. It also appears that the random variable should have a density, though we have no result towards that direction at the moment.

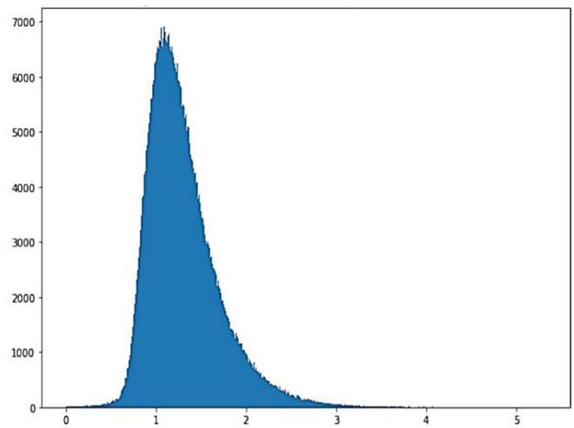

Fig. 5: Limiting distribution for Critical Connectivity distance with $\alpha = 0.5$.

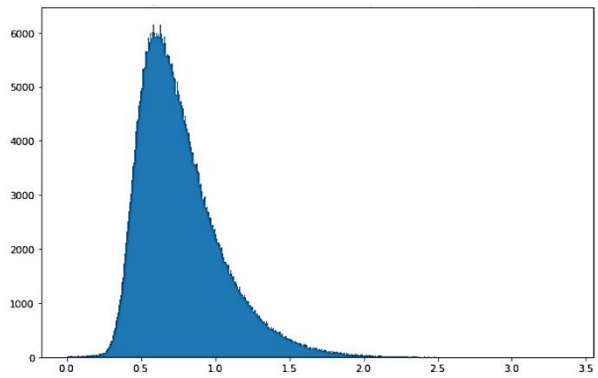

Fig. 6: Limiting distribution for Critical Connectivity distance with $\alpha = 1$.

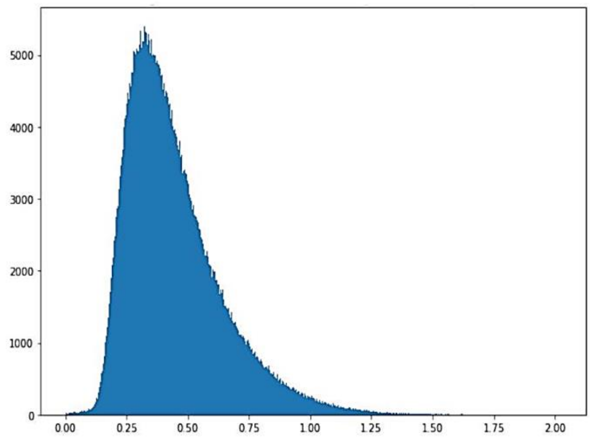

Fig. 7: Limiting distribution for Critical Connectivity distance with $\alpha = 2$.

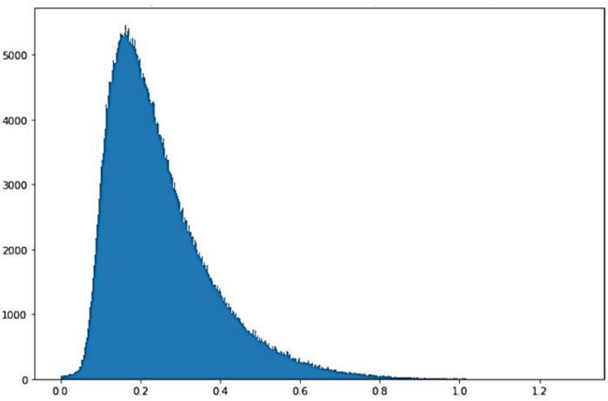

Fig. 8: Limiting distribution for Critical Connectivity distance with $\alpha = 4$.

The theorem clearly shows for large n, the critical value is proportional to inverse power of n, however the limiting random variable is random. In case of Penrose (2003), it is assumed that the densities are bounded away from 0, i.e., there is a constant $c > 0$ such that $f(x) \geq c$ for all $x \in [0,1]$. Clearly, this condition does not hold in our case and hence similar results cannot be expected. Indeed, we show that in our case, the scaling factor is different and we obtain a limit which depends upon the parameter α. Further, in case of Penrose (2003), the limit is non-random while in our case the limit is random.

In order to study the critical parameter $r_c(n: f_\alpha)$, we will introduce another object. Let $X_{(1:n)} < X_{(2:n)} \cdots < X_{(n:n)}$ be the order statistic obtained from

$X_1, X_2, \ldots, X_n$. We define the spacings (distance between the ordered random variables) as follows: for $i = 1,2, \ldots, n-1$, set

$$D_{(i:n)} = X_{(i+1:n)} - X_{(i:n)} \tag{1.6}$$

Let us set

$$D_n = \max\{D_{(i:n)}: 1 \leq i \leq n-1\} \tag{1.7}$$

Let us consider a value r for which graph is connected, then we must have $D_{(i:n)} \leq r$ for all $1 \leq i \leq n-1$. Hence, we have

$$D_n = \max\{D_{(i:n)}: 1 \leq i \leq n-1\} \leq r$$

Hence, $D_n \geq r_c(n; f_\alpha)$

Conversely, if $r < D_n$ then there exists j such that $D_{j:n} > r$. Hence, the distance between $X_{(j+1:n)}$ and $X_{(j:n)}$ must be larger than r and there are no points in between them. So, the pair $\left(X_{(j+1:n)}, X_{(j:n)}\right)$ will not be connected. Thus, the graph is disconnected. Hence, $D_n \leq r_c(n; f_\alpha)$.

So, combining both the inequalities, we observe that the critical value $r_c(n; f_\alpha)$ for the graph to be connected is actually given by

$$r_c(n; f_\alpha) = D_n = \max\{D_{(i:n)}: 1 \leq i \leq n-1\} \tag{1.8}$$

In view of equation (1.8), our Theorem can be restated as follows:

Theorem 2: Let $\{X_i: 1 \leq i \leq n\}$ be i.i.d. samples from a distribution having density f_α for $\alpha > 0$. Let D_n represent the maximal spacing as defined in (1.7). Then,

$$n^{1/(1+\alpha)} D_n \Rightarrow Y_{1+\alpha} \tag{1.9}$$

where $Y_{1+\alpha}$ is as defined above.

In the next section we prove Theorem 2. To prove the result, we will use the representation of order statistics through exponential random variables with parameter 1.

2. Proof of the Result

We first show that the random variable, defined in the introduction, is a proper random variable.

Lemma 1: For any $\beta > 1$, the random variable Y_β defined in (1.4) is finite almost surely.

Proof: We note that S_k is a partial sum of i.i.d. random variables with expectation 1. Thus, by strong law of large numbers (see Theorem 22.1. of Billingsley 1986), we have that

$$\lim_{k\to\infty} \frac{S_k}{k} = 1 \text{ almost surely}$$

Consider the function $g(x) = x^{1/\beta}$. Note, $g'(x) = \frac{1}{\beta} x^{1/\beta - 1}$ and clearly it is a `decreasing function for $x > 0$ as $\beta > 1$. Now, we use the Mean Value Theorem, on g with $u > v > 0$ to obtain

$$g(u) - g(v) = (u - v)g'(c) \le (u - v)g'(v)$$

where $c \in (u, v)$ and we have used the fact the g' is a decreasing function.

Now, setting $v = S_k$ and $u = S_{k+1} = S_k + E_{k+1}$, using above, we obtain that

$$S_{k+1}^{1/\beta} - S_k^{1/\beta} \le E_{k+1} \frac{1}{\beta} (S_k)^{1/\beta - 1} = \frac{1}{\beta} \left(\frac{k}{S_k}\right)^{1-1/\beta} \left(\frac{E_{k+1}}{k^{1-1/\beta}}\right) \quad (2.1)$$

We show that the right hand side will converge to 0 almost surely. Assuming it we first show that Y_β is finite almost surely.

Indeed, for every ω, there will exist N (depending on ω), such that $S_{k+1}^{1/\beta} - S_k^{1/\beta} \le 1$ for all $k \ge N$. Thus, the supremum defined above must be dominated by the maximum of 1 and $S_2^{1/\beta} - S_1^{1/\beta}, S_3^{1/\beta} - S_2^{1/\beta}, \dots, S_N^{1/\beta} - S_{N-1}^{1/\beta}$. Hence Y_β is finite almost surely.

To show that the right hand side of (2.1) converges to 0 almost surely, we use standard Borel Cantelli Lemma (see Theorem 4.3. of Billingsley 1986). Note that it is enough to show that the last term converges to 0 almost surely, as the first term $(k/S_k)^{1-1/\beta}$ will converge to 1 almost surely since S_k/k converges to 1. We show for any $\gamma > 0, E_{k+1}/k^\gamma \to 0$ almost surely. Indeed, for any $\epsilon > 0$, we have

$$\mathbb{P}\left(\frac{E_{k+1}}{k^\gamma} > \epsilon\right) = \mathbb{P}(E_{k+1} > k^\gamma \epsilon) = \exp(-k^\gamma \epsilon)$$

Hence, using the fact that the function $x \mapsto \exp(-x^\gamma \epsilon)$ is decreasing, we have

$$\sum_{k=1}^{\infty} \mathbb{P}(E_{k+1} > k^\gamma \epsilon) = \sum_{k=1}^{\infty} \exp(-k^\gamma \epsilon) \le \sum_{k=1}^{\infty} \int_{k-1}^{k} \exp(-x^\gamma \epsilon) dx = \int_0^\infty \exp(-x^\gamma \epsilon) dx$$
$$= \int_0^\infty \exp(-u) \frac{1}{\gamma} (u/\epsilon)^{\frac{1}{\gamma} - 1} du = \frac{1}{\gamma \epsilon^{\frac{1}{\gamma} - 1}} \Gamma(1/\gamma) < \infty$$

(2.2)

where we have used the transformation $x^{\gamma}\epsilon = u$ in converting the integral. This proves the result.

An important corollary would be required in the proof. Let us define

$$M_n(\beta) = \max\left\{S_{k+1}^{1/\beta} - S_k^{1/\beta} : 1 \le k \le n\right\} \tag{2.3}$$

For every ω for which $Y_\beta(\omega) < \infty$, we must have $M_n(\beta)(\omega) \to Y_\beta(\omega)$. Thus, we have following corollary:

Corollary 2.1: For any $\beta > 1$, we have

$$M_n(\beta) \longrightarrow Y_\beta \text{ as } n \longrightarrow \infty \text{ almost surely} \tag{2.4}$$

Next we prove a lemma which will connect order statistic of random variables to the powers of partial sums of exponentials. For that we need a representation of the order statistics from uniform random variables through exponential random variables. Let $U_1, U_2, \ldots$ be a sequence of i.i.d. uniform random variables. Let

$$X_i = (U_i)^{1/(1+\alpha)}$$

for all $i \ge 1$. We note that the sequence $\{X_i : i \ge 1\}$ are i.i.d. The common density is also given by f_α (see equation (1.2)).

Let $\left(U_{(1:n)} < U_{(2:n)} \cdots < U_{(n:n)}\right)$ be the ordered statistic obtained from the first n uniform random variables, $U_1, U_2, \ldots, U_n$. Since $x \to x^{1/(1+\alpha)}$ is a monotone transformation, we must have,

$$X_{(i:n)} = \left(U_{(i:n)}\right)^{1/(1+\alpha)} \tag{2.5}$$

for $i = 1,2, \ldots, n$.

Now, we will use the representation of order statistic from the uniform distribution. Let $\{E_k : k \ge 1\}$ be a sequence of i.i.d. exponential random variables with parameter 1. Let $S_k = \sum_{i=1}^{k} E_i$ for any $k \ge 1$. Then we can represent the order statistic from uniform distribution as ratio of partial sums of exponentials (see David & Nagaraja (2003) for example). In other words,

$$\left(U_{(1:n)}, \ldots, U_{(n:n)}\right) \stackrel{d}{=} \left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right)$$

Here $\stackrel{d}{=}$ stands for the equality in distribution.

By monotonicity of the transformation $x \to x^{1/(1+\alpha)}$, we have

$$\left(U_{(1:n)}^{1/(1+\alpha)},\dots,U_{(n:n)}^{1/(1+\alpha)}\right) \overset{d}{=} \left(\left(\frac{S_1}{S_{n+1}}\right)^{1/(1+\alpha)},\dots,\left(\frac{S_n}{S_{n+1}}\right)^{1/(1+\alpha)}\right)$$
$$\left(X_{(1:n)},\dots,X_{(n:n)}\right) \overset{d}{=} \left(\left(\frac{S_1}{S_{n+1}}\right)^{1/(1+\alpha)},\dots,\left(\frac{S_n}{S_{n+1}}\right)^{1/(1+\alpha)}\right) \tag{2.6}$$

We will now use the equality in distribution in (2.6) to conclude many of our results.

Lemma 2: Fix any $k \geq 1$. We have

$$n^{1/(1+\alpha)}\left(X_{(1:n)},X_{(2:n)},\dots X_{(k:n)}\right) \Rightarrow \left(S_1^{1/(1+\alpha)},S_2^{1/(1+\alpha)},\dots,S_k^{1/(1+\alpha)}\right) \tag{2.7}$$

Here, $\Rightarrow$ represents convergence in distribution.

Proof: Fix any $k \geq 1$. By constant multiplication also, the order statistics will maintain the above distributional equality. So, for the first k variates, we have

$$n^{1/(1+\alpha)}\left(X_{(1:n)},\dots,X_{(k:n)}\right) \overset{d}{=} \left(\left(\frac{nS_1}{S_{n+1}}\right)^{1/(1+\alpha)},\dots,\left(\frac{nS_k}{S_{n+1}}\right)^{1/(1+\alpha)}\right)$$

Now, we let $n \to \infty$ on the right hand side. Observing that $S_n/n \to 1$ almost surely, by the strong law of large numbers, we can now write the right hand side as,

$$\lim_{n\to\infty}\left(\frac{nS_i}{S_{n+1}}\right)^{1/(1+\alpha)} = S_i^{1/(1+\alpha)}\lim_{n\to\infty}\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} = S_i^{1/(1+\alpha)}$$

almost surely, for each $i = 1,2,\dots,k$. Thus, we have

$$\lim_{n\to\infty}\left(\left(\frac{nS_1}{S_{n+1}}\right)^{1/(1+\alpha)},\dots,\left(\frac{nS_k}{S_{n+1}}\right)^{1/(1+\alpha)}\right) \to \left(S_1^{1/(1+\alpha)},\dots,S_k^{1/(1+\alpha)}\right)$$

almost surely. Therefore, the convergence is also in distribution. Hence, we conclude that

$$\left(\left(\frac{nS_1}{S_{n+1}}\right)^{1/(1+\alpha)},\dots,\left(\frac{nS_k}{S_{n+1}}\right)^{1/(1+\alpha)}\right) \Rightarrow \left(S_1^{1/(1+\alpha)},\dots,S_k^{1/(1+\alpha)}\right)$$

Thus, using equality of distribution in (2.6), we have

$$n^{1/(1+\alpha)}\left(X_{(1:n)},\dots,X_{(k:n)}\right) \Rightarrow \left(S_1^{1/(1+\alpha)},\dots,S_k^{1/(1+\alpha)}\right)$$

This completes the proof.

Next, we will connect the spacings with the limiting random variables.

Lemma 3: Fix any $k \geq 1$. We have

$$n^{1/(1+\alpha)}\left(D_{(1:n)}, \dots D_{(k:n)}\right) \Rightarrow \left(S_2^{1/(1+\alpha)} - S_1^{1/(1+\alpha)}, \dots, S_{k+1}^{1/(1+\alpha)} - S_k^{1/(1+\alpha)}\right) \tag{2.8}$$

Proof : We use Lemma 2 and the continuous mapping theorem (see Theorem 25.7. of Billingsley (1986)). Note that the mapping $g: \mathbb{R}^{k+1} \to \mathbb{R}^k$ defined by

$$g(u_1, u_2, \dots, u_{k+1}) = (u_2 - u_1, \dots, u_{k+1} - u_k)$$

is a continuous function. Hence, using Lemma 2, we have

$$g\left(n^{1/(1+\alpha)}\left(X_{(1:n)}, X_{(2:n)}, \dots X_{(k+1:n)}\right)\right) \Rightarrow g\left(S_1^{1/(1+\alpha)}, S_2^{1/(1+\alpha)}, \dots, S_{k+1}^{1/(1+\alpha)}\right),$$

i.e.,

$$n^{1/(1+\alpha)}\left(D_{(1:n)}, \dots D_{(k:n)}\right) \Rightarrow \left(S_2^{1/(1+\alpha)} - S_1^{1/(1+\alpha)}, \dots, S_{k+1}^{1/(1+\alpha)} - S_k^{1/(1+\alpha)}\right)$$

This proves the result.

Next we will again use continuity theorem to connect with the maximum of the spacings and maximum of the limiting random variables in the above Lemma.

Lemma 4: Fix any $k \geq 1$. We have

$$n^{1/(1+\alpha)}\max\left\{D_{(i:n)}: 1 \leq i \leq k\right\} \Rightarrow M_k(1+\alpha) \tag{2.9}$$

as $n \to \infty$ where $M_k(1+\alpha)$ is as defined in equation (2.3).

Proof : Here we use the fact that the mapping $g: \mathbb{R}^k \to \mathbb{R}$ defined by

$$g(u_1, u_2, \dots, u_k) = \max\{u_i: 1 \leq i \leq k\}$$

is also a continuous function. Hence, applying the continuous mapping Theorem (Theorem 25.7. of Billingsley (1986)) on the result of Lemma 3, we have

$$g\left(n^{1/(1+\alpha)}\left(D_{(1:n)}, \dots D_{(k:n)}\right)\right) \Rightarrow g\left(\left(S_2^{1/(1+\alpha)} - S_1^{1/(1+\alpha)}, \dots, S_{k+1}^{1/(1+\alpha)} - S_k^{1/(1+\alpha)}\right)\right)$$

i.e.,

$$\begin{aligned} n^{1/(1+\alpha)}\max\left\{D_{(i:n)}: 1 \leq i \leq k\right\} &= \max\left\{n^{1/(1+\alpha)}D_{(i:n)}: 1 \leq i \leq k\right\} \\ &\Rightarrow \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: 1 \leq i \leq k\right\} \\ &= M_k(1+\alpha) \end{aligned}$$

This proves the result.

Before we prove the result, we prove one more auxiliary result which we will use in the proof.

Lemma 5: For any $x > 0$, we have

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(n^{1/(1+\alpha)}\max\left\{D_{(i:n)}: k+1 \le i \le n-1\right\} > x\right) = 0 \quad (2.10)$$

Proof : We know from the representation in equation (2.6) that

$$n^{1/(1+\alpha)}\max\left\{D_{(i:n)}: k+1 \le i \le n-1\right\}$$
$$\stackrel{d}{=} \max\left\{\left(\frac{nS_{i+1}}{S_{n+1}}\right)^{1/(1+\alpha)} - \left(\frac{nS_i}{S_{n+1}}\right)^{1/(1+\alpha)} : k+1 \le i \le n-1\right\}.$$

Thus, we have

$$\mathbb{P}\left[n^{1/(1+\alpha)}\max\left\{D_{(i:n)}: k+1 \le i \le n-1\right\} > x\right]$$
$$= \mathbb{P}\left[\max\left\{\left(\frac{nS_{i+1}}{S_{n+1}}\right)^{1/(1+\alpha)} - \left(\frac{nS_i}{S_{n+1}}\right)^{1/(1+\alpha)} : k+1 \le i \le n-1\right\} > x\right]$$
$$= \mathbb{P}\left[\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \le i \le n-1\right\} > x\right].$$

We will split the above event into two parts by intersecting with the events $\{S_{n+1} \le n/2\}$ and $\{S_{n+1} > n/2\}$. Thus, we have

$$\mathbb{P}\left[\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \le i \le n-1\right\} > x\right]$$
$$= \mathbb{P}\left[\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \le i \le n-1\right\} > x, S_{n+1} > n/2\right]$$
$$+\mathbb{P}\left[\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \le i \le n-1\right\} > x, S_{n+1} \le n/2\right]$$
$$\le \mathbb{P}\left[\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \le i \le n-1\right\} > x, S_{n+1} > n/2\right]$$
$$+\mathbb{P}(S_{n+1} \le n/2).$$

The second term is easy to estimate. We have

$$\mathbb{P}(S_{n+1} \le n/2) = \mathbb{P}(S_{n+1} - n - 1 \le -1 - n/2)$$
$$\le \mathbb{P}(|S_{n+1} - n - 1| \ge 1 + n/2).$$

Using Chebyshev's inequality, we have

$$\mathbb{P}(S_{n+1} \le n/2) \le \frac{\mathrm{Var}(S_{n+1})}{(n+2)^2/4} = \frac{4(n+1)}{(n+2)^2}$$

We observe that $\mathbb{P}(S_{n+1} \leq n/2) \to 0$ as $n \to \infty$

For the first term, we note that if $S_{n+1} > n/2$, then $\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} < 2^{1/(1+\alpha)}$. Hence, we have $\max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \leq i \leq n-1\right\} > x/2^{1/(1+\alpha)}$. Thus, we have that

$$\left\{\left(\frac{n}{S_{n+1}}\right)^{1/(1+\alpha)} \max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \leq i \leq n-1\right\} > x, S_{n+1} > n/2\right\}$$
$$\subseteq \left\{\max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \leq i \leq n-1\right\} > x/2^{1/(1+\alpha)}\right\}.$$

Let us denote $1/2^{1/(1+\alpha)}$ by c. Further, we see that the event in the right hand side can be written as

$$\left\{\max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)}: k+1 \leq i \leq n-1\right\} > cx\right\}$$
$$\subseteq \bigcup_{i=k+1}^{n-1} \left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)} > cx\right\}$$

Again, using the same estimate as in (2.1), we have

$$S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)} \leq \frac{E_{i+1}}{(1+\alpha)S_i^{\alpha/(1+\alpha)}}$$

So, we have that

$$\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)} > cx\right\} \subseteq \left\{\frac{E_{i+1}}{(1+\alpha)S_i^{\alpha/(1+\alpha)}} > cx\right\}$$

We will split this event into two more parts. We have

$$\left\{\frac{E_{i+1}}{(1+\alpha)S_i^{\alpha/(1+\alpha)}} > cx\right\}$$
$$= \left\{\frac{E_{i+1}}{(1+\alpha)S_i^{\alpha/(1+\alpha)}} > cx, S_i \leq i/2\right\} \cup \left\{\frac{E_{i+1}}{(1+\alpha)S_i^{\alpha/(1+\alpha)}} > cx, S_i > i/2\right\}$$
$$\subseteq \{S_i \leq i/2\} \cup \left\{E_{i+1} > (1+\alpha)i^{\alpha/(1+\alpha)}cx\right\}$$

where the last inclusion follows from the fact that if $S_i > i/2$, then we must have $E_{i+1} > (1+\alpha)i^{\alpha/(1+\alpha)}cx$. Combining all these, we obtain that

$$\begin{aligned}
&\mathbb{P}\left[\max\left\{S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)} : k+1 \le i \le n-1\right\} > cx\right] \\
&\le \sum_{i=k+1}^{n-1} \mathbb{P}\left[S_{i+1}^{1/(1+\alpha)} - S_i^{1/(1+\alpha)} > cx\right] \\
&\le \sum_{i=k+1}^{n-1} \mathbb{P}[S_i \le i/2] + \sum_{i=k+1}^{n-1} \mathbb{P}\left[E_{i+1} > (1+\alpha) i^{\alpha/(1+\alpha)} cx\right] \\
&\le \sum_{i=k+1}^{\infty} \mathbb{P}[S_i \le i/2] + \sum_{i=k+1}^{\infty} \mathbb{P}\left[E_{i+1} > (1+\alpha) i^{\alpha/(1+\alpha)} cx\right].
\end{aligned}$$

Thus, we have obtained,

$$\begin{aligned}
&\mathbb{P}\left[n^{1/(1+\alpha)} \max\left\{D_{(i:n)} : k+1 \le i \le n-1\right\} > x\right] \\
&\le \frac{4(n+1)}{(n+2)^2} + \sum_{i=k+1}^{\infty} \mathbb{P}[S_i \le i/2] + \sum_{i=k+1}^{\infty} \mathbb{P}\left[E_{i+1} > (1+\alpha) i^{\alpha/(1+\alpha)} cx\right].
\end{aligned}$$

Taking limsup as n increases to infinity, we see that the first term in the right hand side converges to 0 while the remaining two terms are independent of n. So, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\left[n^{1/(1+\alpha)} \max\left\{D_{(i:n)} : k+1 \le i \le n-1\right\} > x\right] \\
&\le \sum_{i=k+1}^{\infty} \mathbb{P}[S_i \le i/2] + \sum_{i=k+1}^{\infty} \mathbb{P}\left[E_{i+1} > (1+\alpha) i^{\alpha/(1+\alpha)} cx\right].
\end{aligned}$$

Now, note that we will need to take a further limsup as k increases to infinity. Further the above sums are the tail parts of the sums $\sum_{i=1}^{\infty} \mathbb{P}[S_i \le i/2]$ and $\sum_{i=1}^{\infty} \mathbb{P}\left[E_{i+1} > (1+\alpha) i^{\alpha/(1+\alpha)} cx\right]$ respectively. If we are able to show that the sums are finite, then the tail part will converge to 0, proving our result.

For the first sum, we will employ a similar method, except the fact that we need to go to higher moment this time. Note that S_i follows a Gamma distribution with parameter i and 1. Hence the fourth centered moment is given by $6i + 3i^2 \le 9i^2$. Thus, we have

$$\mathbb{P}[S_i \le i/2] \le \mathbb{P}[|S_i - i| \ge i/2] \le \frac{\mathbb{E}((S_i - i)^4)}{(i/2)^4} \le \frac{9i^2}{i^4/16} = \frac{144}{i^2}.$$

This takes care of the first summation.

For the second summation, we have already observed that (see equation (2.2)), for any $\gamma > 0$ and $\epsilon > 0$, the sum $\sum_{i=1}^{\infty} \mathbb{P}(E_{i+1} > \epsilon i^{\gamma}) < \infty$. Therefore, the sum is also finite. This completes the proof.

Now, we are ready to prove our result. Let us introduce the following notation for the ease of writing the steps of calculations:

$$Z_n(i,j) := n^{1/(1+\alpha)} \max\{D_{(k:n)} : i \le k \le j\} \tag{2.11}$$

Note that we may rewrite Lemma 4 and Lemma 5 in terms of above notation. Indeed, we have, $Z_n(1,k) \Rightarrow M_k(1+\alpha)$ as $n \to \infty$. Similarly, we can rewrite, to say $\limsup_{k_0\to\infty} \limsup_{n\to\infty} \mathbb{P}(Z_n(k_0+1, n-1) > x) = 0$.

Proof of Theorem 2 : Fix any $x > 0$. We want to prove that

$$\begin{aligned}\lim_{n\to\infty} \quad & \mathbb{P}\left(n^{1/(1+\alpha)} \max\{D_{(i:n)} : 1 \le i \le n-1\} \le x\right) \\ &= \lim_{n\to\infty} \mathbb{P}(Z_n(1,n-1) \le x) = \mathbb{P}(Y_{1+\alpha} \le x).\end{aligned} \tag{2.12}$$

First, we observe that, for any fixed $k_0 < n$, we must have

$$\{Z_n(1,k_0) \le x\} \subseteq \{Z_n(1,n-1) \le x\}$$

as in the right hand side, we have introduced more conditions. Thus, we have

$$\mathbb{P}(Z_n(1,n-1) \le x) \le \mathbb{P}(Z_n(1,k_0) \le x)$$

Taking limsup of both sides and observing that the limit of the right hand exists by Lemma 4, we obtain that

$$\limsup_{n\to\infty} \mathbb{P}(Z_n(1,n-1) \le x) \le \lim_{n\to\infty} \mathbb{P}(Z_n(1,k_0) \le x) = \mathbb{P}\left(M_{k_0}(1+\alpha) \le x\right)$$

Now, we may let k_0 increase to infinity. By Corollary 2.1, $M_{k_0}(1+\alpha)$ converges almost surely to $Y_{1+\alpha}$; hence in distribution. Now, observing that the left hand side does not depend on k_0, we have that

$$\limsup_{n\to\infty} \mathbb{P}(Z_n(1,n-1) \le x) \le \lim_{k_0\to\infty} \mathbb{P}\left(M_{k_0}(1+\alpha) \le x\right) = \mathbb{P}(Y_{1+\alpha} \le x) \tag{2.13}$$

For the other bound, we again fix $k_0 < n$. We have that

$$
\begin{aligned}
&\mathbb{P}(Z_n(1,k_0) \le x) \\
&= \mathbb{P}(Z_n(1,k_0) \le x, Z_n(1,n-1) \le x) + \mathbb{P}(Z_n(1,k_0) \le x, Z_n(1,n-1) > x) \\
&= \mathbb{P}(Z_n(1,n-1) \le x) + \mathbb{P}(Z_n(1,k_0) \le x, Z_n(k_0+1,n-1) > x) \\
&\le \mathbb{P}(Z_n(1,n-1) \le x) + \mathbb{P}(Z_n(k_0+1,n-1) > x).
\end{aligned}
$$

Thus, we have

$$\mathbb{P}(Z_n(1,n-1) \le x) \ge \mathbb{P}(Z_n(1,k_0) \le x) - \mathbb{P}(Z_n(k_0+1,n-1) > x) \quad (2.14)$$

We take liminf of both sides of (2.14). Again, from Lemma 4, we obtain that limit of the first term in the right hand side exists and equals $\mathbb{P}\big(M_{k_0}(1+\alpha) \le x\big)$. Also, noting that $\text{liminf}(a_n - b_n) \ge \text{liminf}a_n - \text{limsup}b_n$, we obtain,

$$
\begin{aligned}
&\liminf_{n\to\infty}\mathbb{P}(Z_n(1,n-1) \le x) \\
&\ge \liminf_{n\to\infty}\mathbb{P}(Z_n(1,k_0) \le x) - \limsup_{n\to\infty}\mathbb{P}(Z_n(k_0+1,n-1) > x) \\
&= \mathbb{P}\big(M_{k_0}(1+\alpha) \le x\big) - \limsup_{n\to\infty}\mathbb{P}(Z_n(k_0+1,n-1) > x).
\end{aligned}
$$

Now again we take liminf as $k_0 \to \infty$. Since the left hand side is independent of k_0 and $M_{k_0}(1+\alpha)$ converges almost surely to $Y_{1+\alpha}$, we have

$$
\begin{aligned}
&\liminf_{n\to\infty}\mathbb{P}(Z_n(1,n-1) \le x) \\
&\ge \liminf_{k_0\to\infty}\mathbb{P}\big(M_{k_0}(1+\alpha) \le x\big) - \limsup_{k_0\to\infty}\limsup_{n\to\infty}\mathbb{P}(Z_n(k_0+1,n-1) > x) \\
&= \mathbb{P}(Y_{1+\alpha} \le x)
\end{aligned}
$$

where in the last line we have used Lemma 5 to conclude that the last term has no contribution. Therefore, combining with equation (2.13), we conclude the result in equation (2.12). This proves the result.

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