ISSN 0971-0388

Aligarh Journal of Statistics
Vol. 43(2023), 1-18

Estimation of the Location Parameter of Some Distributions Using Extreme Ranked Set Sampling with Known Coefficient of Variation

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[Received on January, 2021. Accepted on February, 2023]

ABSTRACT

In this work we have considered the problem of estimation of the location parameter of some distributions using extreme ranked set sampling (ERSS) with known coefficient of variation. We have discussed the general theory of estimation of the location parameter of a location-scale family of distributions, when the scale parameter is proportional to the location parameter using ERSS. Also we have estimated the best linear unbiased estimator (BLUE) of the mean of the normal distribution, logistic distribution and double exponential distributions with known coefficient of variation d using ERSS for some specific values of the sample size n.

1. Introduction

The ranked set sampling (RSS) can be applied in many areas such as forest, agriculture, animal science, medicine etc. For details about these applications see, Halls and Dell (1966) and Al-saleh and Al-Sharafat (2001). Takahasi and Wakimoto (1968) established rigorous statistical foundation on the theory of RSS. Samawi *et al.* (1996) used extreme ranked set sampling (ERSS), when RSS based on extremes for both even and odd sample sizes. In practice, ranking a sample of moderate size and observing the i^{th} ranked unit (ranking of middle ordered units) is a difficult task but it is not difficult to identify maximum or minimum units. Thus ERSS is a better transformation than RSS. In ERSS we can

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increase the set size with reduced ranking errors. The ERSS from a location scale family of distribution is described below.
Consider the location scale family of distributions of the form

$$h(z;\psi,\lambda)=\frac{1}{\lambda}h_0\left(\frac{z-\psi}{\lambda}\right),\psi\in R\,,\ \lambda>0\,,\,z\in R\,, \qquad (1)$$

where the form h_0 is known .Suppose we take n sets of observations from the pdf of the form given in (1) and each set contain n observations. Then the ESSR from the distribution of the form (1) is described below.

Case 1 : When n is even .

Suppose n=2m, where m is any positive integer. Let $Z_{(1:n)r}$ denote the first order statistic from the r^{th} sample of size n, r=1,2,…,m arising from (1). Also let $Z_{(n::n)s}$ denote the n^{th} order statistic from the s^{th} sample of size n, s=m+1, m+2, …,2m. Now $Z_{(1:n)1},Z_{(1:n)2},...,Z_{(1:n)m},Z_{(n::n)m+1},Z_{(n::n)m+2},...,Z_{(n::n)2m}$ forms ERSS arising from (1).

Case 2: when n is odd:

Suppose n=2m+1, where m is any positive integer. Let $Z_{(1:n)r}$ denote the first order statistic from the r^{th} sample r=1,2,…,m+1 arising from (1).Also let $Z_{(n:n)s}$ denote n^{th} order statistic from the s^{th} sample , s=m+2,m+3,…,2m+1. Now $Z_{(1:n)1},Z_{(1:n)2},\ldots Z_{(1:n)m+1`},Z_{(n:n)m+2},\ldots Z_{(n:n)2m+1}$ forms ERSS arising from (1).

Lesitha and Thomas (2013) discussed the estimation of parameters of a location-scale family of distributions using ERSS.

The problem of estimating the location parameter of a distribution when scale parameter is proportional to the location parameter are reported in the literature see, Glesser and Healy (1976), Searls (1964), Khan (1968), Arnholt and Hebert (1995), Kunte (2000), Guo and Pal (2003), Thomas and Sajeevkumar (2003), Sajeevkumar and Thomas (2005) and Sajeevkumar and Irshad (2013). Estimation of the location parameter of a location-scale family of distributions when the scale parameter is proportional to the location parameter using RSS are discussed by Irshad and Sajeevkumar (2011). Hence in this work our aim is to study the problem of estimation of the location parameter of some distributions with known coefficient of variation by ERSS.

2. Estimation of the Location Parameter of a Distribution When the Scale Parameter is Proportional to the Location Parameter Using ERSS.

In this section, we consider the location-scale family of distributions which depend on a location parameter ψ (>0) and a scale parameter λ, such that $\lambda = d\psi$, where d is known with pdf given by

$$h(z;\psi,d\psi) = \frac{1}{d\psi} h_0\left(\frac{z-\psi}{d\psi}\right), \psi > 0\,, d > 0\,, z \in R, \qquad (2)$$

2.1 Estimation of tahe Location Parameter ψ When n is Even Using ERSS.

Suppose n=2m, where m is any positive integer. Let $Z_{(1:n)r}$ denote the first order statistic from the r^{th} sample of size n, r=1,2,…,m arising from (2). Also let $Z_{(n::n)s}$ denote the n^{th} order statistic from the s^{th} sample of size n , s=m+1, m+2,…,2m .Now $Z_{(1:n)1}, Z_{(1:n)2},..., Z_{(1:n)m}, Z_{(n::n)m+1}, Z_{(n::n)m+2},..., Z_{(n::n)2m}$ forms ERSS arising from (2).Let $\underline{Z} = \left(Z_{(1:n)1}, Z_{(1:n)2},..., Z_{(1:n)m}, Z_{(n::n)m+1}, Z_{(n::n)m+2},..., Z_{(n::n)2m}\right)'$ be the vector of ERSS arising from (2). Define $\underline{W} = \left(W_{(1:n)1}, W_{(1:n)2},..., W_{(1:n)m}, W_{(n::n)m+1}, W_{(n::n)m+2},..., W_{(n::n)2m}\right)'$ as a vector of corresponding observations in an ERSS arising from $h(x,0,1)$. Let $\underline{\gamma} = \left(\gamma_{(1:n)1}, \gamma_{(1:n)2},..., \gamma_{(1:n)m}, \gamma_{(n::n)m+1}, \gamma_{(n::n)m+2},..., \gamma_{(n::n)2m}\right)'$ be the vector of means and $C = D(\underline{W}) = ((\beta_{ij:n}))$ be the dispersion matrix of $\underline{W}$. Clearly $C = dig\left(c_{(1:n)1}, c_{(1:n)2},.., c_{(1:n)m}, c_{(n:n)m+1}, c_{(n:n)m+2},..., c_{(n:n)2m}\right)$, $c_{(i:n)j} = \beta_{i,i:n}$, for i=1,n;j=1,2,…,n. Then considering ψ as the location parameter of (2), a linear unbiased estimator of ψ based on ESSR is given by (see Lam *et al.* (1994), p.726)

$$\hat{\psi} = \frac{\left[\sum_{i=1}^{m}\frac{Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{Z_{(n:n)i}}{c_{(n:n)i}}\right]\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}^{2}}{c_{(n:n)i}}\right]}{\left[\sum_{i=1}^{m}\frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{1}{c_{(n:n)i}}\right]\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}^{2}}{c_{(n:n)i}}\right] - \left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right]^{2}} -$$

$$\frac{\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n::n)i}}{c_{(n::n)i}}\right]\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}Z_{(n:n)i}}{c_{(n:n)i}}\right]}{\left[\sum_{i=1}^{m}\frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{1}{c_{(n::n)i}}\right]\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n::n)i}^{2}}{c_{(n::n)i}}\right] - \left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n::n)i}}{c_{(n::n)i}}\right]^{2}} \qquad (3)$$

and

$$Var(\hat{\psi}) = \frac{\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n::n)i}^{2}}{c_{(n::n)i}}\right]d^{2}\psi^{2}}{\left[\sum_{i=1}^{m}\frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{1}{c_{(n::n)i}}\right]\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n::n)i}^{2}}{c_{(n::n)i}}\right] - \left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n::n)i}}{c_{(n::n)i}}\right]^{2}} \qquad (4)$$

Now we derive the BLUE of ψ involved in (2) using ERSS, when the sample size n is even is given in the following theorem.

Theorem 2.1

Let $\underline{Z} = \left(Z_{(1:n)1}, Z_{(1:n)2}, ..., Z_{(1:n)m}, Z_{(n::n)m+1}, Z_{(n::n)m+2}, ..., Z_{(n::n)2m}\right)'$ be the vector of ERSS arising from (2). Define $\underline{W} = \left(W_{(1:n)1}, W_{(1:n)2}, ..., W_{(1:n)m}, W_{(n::n)m+1}, W_{(n::n)m+2}, ..., W_{(n::n)2m}\right)'$ as a vector of corresponding observations in an ERSS arising from $h(x,0,1)$.Let $\underline{\gamma} = \left(\gamma_{(1:n)1}, \gamma_{(1:n)2}, ..., \gamma_{(1:n)m}, \gamma_{(n:n)m+1}, \gamma_{(n:n)m+2}, ..., \gamma_{(n:n)2m}\right)'$ be the vector of means and $C = D(\underline{W}) = \left(\left(\beta_{ij:n}\right)\right)$ be the dispersion matrix of $\underline{W}$. Clearly $C = dig(c_{(1:n)1}, c_{(1:n)2}, ..., c_{(1:n)m}, c_{(n:n)m+1}, c_{(n:n)m+2}, ..., c_{(n:n)2m}),$ where $c_{(i:n)j} = \beta_{i,i:n}$, for i=1, n ; j=1,2,…,n. Then BLUE of ψ , say $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{d\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}Z_{(n:n)i}}{c_{(n:n)i}}\right] + \left[\sum_{i=1}^{m}\frac{Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{Z_{(n:n)i}}{c_{(n:n)i}}\right]}{\left[d^2\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^2}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}^2}{c_{(n:n)i}}\right] + 2d\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right] + \left[\sum_{i=1}^{m}\frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{1}{c_{(n:n)i}}\right]} \tag{5}$$

and

$$Var(\tilde{\psi}) = \frac{d^2\psi^2}{\left[d^2\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}^2}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}^2}{c_{(n:n)i}}\right] + 2d\left[\sum_{i=1}^{m}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right] + \left[\sum_{i=1}^{m}\frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n}\frac{1}{c_{(n:n)i}}\right]} \tag{6}$$

Proof: Let $\underline{Z} = \left(Z_{(1:n)1}, Z_{(1:n)2}, \ldots, Z_{(1:n)m}, Z_{(n:n)m+1}, Z_{(n:n)m+2}, \ldots, Z_{(n:n)2m}\right)'$ be the vector of ERSS arising from (2). Define $\underline{W} = \left(W_{(1:n)1}, W_{(1:n)2}, \ldots, W_{(1:n)m}, W_{(n:n)m+1}, W_{(n:n)m+2}, \ldots, W_{(n:n)2m}\right)'$ as a vector of corresponding observations in an ERSS arising from $h(x,0,1)$.Let $\underline{\gamma} = \left(\gamma_{(1:n)1}, \gamma_{(1:n)2}, \ldots, \gamma_{(1:n)m}, \gamma_{(n:n)m+1}, \gamma_{(n:n)m+2}, \ldots, \gamma_{(n:n)2m}\right)'$ be the vector of means and $C = D(\underline{W}) = ((\beta_{ij:n}))$ be the dispersion matrix of $\underline{W}$. Clearly $C = dig(c_{(1:n)1}, c_{(1:n)2}, \ldots, c_{(1:n)m}, c_{(n:n)m+1}, c_{(n:n)m+2}, \ldots, c_{(n:n)2m})$, where $c_{(i:n)j} = \beta_{i,i:n}$ for $i = 1, n;\ j = 1, 2, \ldots, n.$

Clearly

$$E(\underline{Z}) = \psi\underline{1} + d\psi\underline{\gamma} = (d\underline{\gamma} + \underline{1})\psi \tag{7}$$

and

$$D(\underline{Z}) = Cd^2\psi^2\ , \tag{8}$$

where $\underline{1}$ is a column vector of n ones. Equations (7) and (8) together defines a generalized Gauss-Markov set up and hence the BLUE $\tilde{\psi}$ of ψ is obtained as,

$$\tilde{\psi} = \frac{\left(d\underline{\gamma} + \underline{1}\right)' C^{-1}}{\left(d\underline{\gamma} + \underline{1}\right)' C^{-1} \left(d\underline{\gamma} + \underline{1}\right)} \underline{Z} \tag{9}$$

And

$$Var(\tilde{\psi}) = \frac{d^2 \psi^2}{\left(d\underline{\gamma} + \underline{1}\right)' C^{-1} \left(d\underline{\gamma} + \underline{1}\right)} \tag{10}$$

Now we have the following results based on extreme ranked set sample,

$$\underline{1}' C^{-1} \underline{1} = \left[\sum_{i=1}^{m} \frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{1}{c_{(n:n)i}} \right] \quad , \underline{\gamma}' C^{-1} \underline{\gamma} = \left[\sum_{i=1}^{m} \frac{\gamma^2_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma^2_{(n:n)i}}{c_{(n:n)i}} \right],$$

$$\underline{\gamma}' C^{-1} \underline{1} = \left[\sum_{i=1}^{m} \frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma_{(n:n)i}}{c_{(n:n)i}} \right], \ \underline{1}' C^{-1} \underline{Z} = \left[\sum_{i=1}^{m} \frac{Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{Z_{(n:n)i}}{c_{(n:n)i}} \right]$$

And

$$\underline{\gamma}' C^{-1} \underline{Z} = \left[\sum_{i=1}^{m} \frac{\gamma_{(1:n)i} Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma_{(n:n)i} Z_{(n:n)i}}{c_{(n:n)i}} \right]$$

.Now using the above results, (9) and (10) reduces to

$$\tilde{\psi} = \frac{d\left[\sum_{i=1}^{m} \frac{\gamma_{(1:n)i} Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma_{(n:n)i} Z_{(n:n)i}}{c_{(n:n)i}} \right] + \left[\sum_{i=1}^{m} \frac{Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{Z_{(n:n)i}}{c_{(n:n)i}} \right]}{\left[d^2 \sum_{i=1}^{m} \frac{\gamma^2_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma^2_{(n:n)i}}{c_{(n:n)i}} \right] + 2d\left[\sum_{i=1}^{m} \frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma_{(n:n)i}}{c_{(n:n)i}} \right] + \left[\sum_{i=1}^{m} \frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{1}{c_{(n:n)i}} \right]}$$

and

$$Var(\tilde{\psi}) = \frac{d^2 \psi^2}{\left[d^2 \sum_{i=1}^{m} \frac{\gamma^2_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma^2_{(n:n)i}}{c_{(n:n)i}} \right] + 2d\left[\sum_{i=1}^{m} \frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{\gamma_{(n:n)i}}{c_{(n:n)i}} \right] + \left[\sum_{i=1}^{m} \frac{1}{c_{(1:n)i}} + \sum_{i=m+1}^{n} \frac{1}{c_{(n:n)i}} \right]}.$$

Thus the theorem is proved.

2.2 Estimation of the Location Parameter ψ when n is Odd Using ERSS

Suppose n=2m+1, where m is any positive integer. Let $Z_{(1:n)r}$ denote the first order statistic from the r^{th} sample of size n , r=1,2,…,m+1 arising from (2). Also let $Z_{(n:n)s}$ denote n^{th} order statistic from the s^{th} sample of size n, s=m+2, m+3,…,2m+1.Now $Z_{(1:n)1}, Z_{(1:n)2},..., Z_{(1:n)m+1}, Z_{(n:n)m+2},..., Z_{(n:n)2m+1}$ forms ERSS arising from (2). $\underline{Z} = \left(Z_{(1:n)1}, Z_{(1:n)2},..., Z_{(1:n)m+1}, Z_{(n:n)m+2},..., Z_{(n:n)2m+1}\right)'$ be the vector of ERSS arising from (2). Define $\underline{W} = \left(W_{(1:n)1}, W_{(1:n)2},..., W_{(1:n)m+1}, W_{(n:n)m+2}, W_{(n:n)m+3},..., W_{(n:n)2m+1}\right)'$ as a vector of corresponding observations in an ERSS from $h(x,0,1)$.Let $\underline{\gamma} = \left(\gamma_{(1:n)1}, \gamma_{(1:n)2},..., \gamma_{(1:n)m+1}, \gamma_{(n:n)m+2}, \gamma_{(n:n)m+3},..., \gamma_{(n:n)2m+1}\right)'$ be the vector of means and $C = D(\underline{W}) = ((\beta_{ij:n}))$ be the dispersion matrix of $\underline{W}$. Clearly $C = dig\left(c_{(1:n)1}, c_{(1:n)2},..., c_{(1:n)m+1}, c_{(n:n)m+2}, c_{(n:n)m+3},..., c_{(n:n)2m+1}\right)$, where $c_{(i:n)j} = \beta_{i,i:n}$ for
$i = 1, n;\ j = 1,2,..., n.$

Then considering ψ as the location parameter of (2), a linear unbiased estimator of ψ based on ESSR is given by (see Lam *et al.* (1994), p.726)

$$\hat{\psi} = \frac{\left[\sum_{i=1}^{m+1}\frac{Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{Z_{(n:n)i}}{c_{(n:n)i}}\right]\left[\sum_{i=1}^{m+1}\frac{\gamma^2_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma^2_{(n:n)i}}{c_{(n:n)i}}\right]}{\left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{1}{c_{(n:n)i}}\right]\left[\sum_{i=1}^{m+1}\frac{\gamma^2_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma^2_{(n:n)i}}{c_{(n:n)i}}\right] - \left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right]^2}$$

$$\frac{\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n::n)i}}{c_{(n::n)i}}\right]\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}Z_{(n:n)i}}{c_{(n:n)i}}\right]}{\left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{1}{c_{(n::n)i}}\right]\left[\sum_{i=1}^{m+1}\frac{\gamma^2_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma^2_{(n::n)i}}{c_{(n::n)i}}\right] - \left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n::n)i}}{c_{(n::n)i}}\right]^2} \quad (11)$$

and

$$Var(\hat{\psi}) = \frac{\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^2}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n::n)i}^2}{c_{(n::n)i}}\right] d^2\psi^2}{\left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{1}{c_{(n::n)i}}\right]\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^2}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n::n)i}^2}{c_{(n::n)i}}\right] - \left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n::n)i}}{c_{(n::n)i}}\right]^2}. \tag{12}$$

Now we derive the BLUE of ψ involved in (2) using ERSS, when the sample size n is odd is given in the following theorem.

Theorem 2.2

Let $\underline{Z} = \left(Z_{(1:n)1}, Z_{(1:n)2}, \ldots, Z_{(1:n)m+1}, Z_{(n:n)m+2}, Z_{(n:n)m+3}, \ldots, Z_{(n:n)2m+1}\right)'$ be the vector of ERSS arising from (2). Define $\underline{W} = \left(W_{(1:n)1}, W_{(1:n)2}, \ldots, W_{(1:n)m+1}, W_{(n::n)m+2}, W_{(n:n)m+3}, \ldots, W_{(n:n)2m+1}\right)'$ as a vector of corresponding observations in an ERSS arising from $h(z,0,1)$. Let $\underline{\gamma} = \left(\gamma_{(1:n)1}, \gamma_{(1:n)2}, \ldots, \gamma_{(1:n)m+1}, \gamma_{(n:n)m+2}, \gamma_{(n:n)m+3}, \ldots, \gamma_{(n:n)2m+1}\right)'$ be the vector of means and $C = D(\underline{W}) = \left(\left(\beta_{ij:n}\right)\right)$ be the dispersion matrix of $\underline{W}$. Clearly $C = dig(c_{(1:n)1}, c_{(1:n)2}, \ldots, c_{(1:n)m+1}, c_{(n:n)m+2}, c_{(n:n)m+3}, \ldots, c_{(n:n)2m+1}),$ where $c_{(i:n)j} = \beta_{i,i:n}$, for i=1,n;j=1,2,…,2m+1.Then BLUE of ψ , say $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{d\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}Z_{(n:n)i}}{c_{(n:n)i}}\right] + \left[\sum_{i=1}^{m+1}\frac{Z_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{Z_{(n:n)i}}{c_{(n:n)i}}\right]}{\left[d^2\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^2}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}^2}{c_{(n:n)i}}\right] + 2d\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right] + \left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{1}{c_{(n:n)i}}\right]} \tag{13}$$

and

$$Var(\tilde{\psi}) = \frac{d^2\psi^2}{\left[d^2\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^2}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}^2}{c_{(n:n)i}}\right] + 2d\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right] + \left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}} + \sum_{i=m+2}^{n}\frac{1}{c_{(n:n)i}}\right]} \tag{14}$$

Proof: Let $\underline{Z}^* = \left(Z_{(1:n)1}, Z_{(1:n)2}, ..., Z_{(1:n)m+1}, Z_{(n:n)m+2}, Z_{(n:n)m+2}, ..., Z_{(n:n)2m+1}\right)'$ be the vector of ERSS arising from (2). Define $\underline{W}^* = \left(W_{(1:n)1}, W_{(1:n)2}, ..., W_{(1:n)m+1}, W_{(n:n)m+2}, W_{(n:n)m+3}, ..., W_{(n:n)2m+1}\right)'$ as a vector of corresponding observations in an ERSS arising from $h(x,0,1)$. Let $\underline{\gamma}^* = \left(\gamma_{(1:n)1}, \gamma_{(1:n)2}, ..., \gamma_{(1:n)m+1}, \gamma_{(n:n)m+2}, \gamma_{(n:n)m+3}, ..., \gamma_{(n:n)2m+1}\right)'$ be the vector of means and $C^* = D\left(\underline{W}^*\right) = \left(\left(\beta_{ij:n}\right)\right)$ be the dispersion matrix of $\underline{Y}^*$. Clearly $C^* = dig\,(c_{(1:n)1}, c_{(1:n)2}, ..., c_{(1:n)m}, c_{(n:n)m+1}, c_{(n:n)m+2}, ..., c_{(n:n)2m})$, where $c_{(i:n)j} = \beta_{i,i:n}$ for $i = 1, n;\ j = 1, 2, ..., n.$

Clearly

$$E\left(\underline{Z}^*\right) = \psi \underline{1} + d\psi \underline{\gamma}^* = \left(d\underline{\gamma}^* + \underline{1}\right)\psi \tag{15}$$

and

$$D\left(\underline{Z}^*\right) = C^* d^2 \psi^2, \tag{16}$$

Where $\underline{1}$ is a column vector of n ones. Equations (15) and (16) together defines a generalized Gauss-Markov set up and hence the BLUE $\tilde{\psi}$ of ψ is obtained as,

$$\tilde{\psi} = \frac{\left(d\underline{\gamma}^* + \underline{1}\right)' C^{*-1}}{\left(d\underline{\gamma}^* + \underline{1}\right)' C^{*-1} \left(d\underline{\gamma}^* + \underline{1}\right)} \underline{Z}^* \tag{17}$$

and

$$Var(\tilde{\psi}) = \frac{d^2 \psi^2}{\left(d\underline{\gamma}^* + \underline{1}\right)' C^{*-1} \left(d\underline{\gamma}^* + \underline{1}\right)} \tag{18}$$

Now we have the following results based on extreme ranked set sample,

$$\underline{1}'C^{*-1}\underline{1}=\left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{1}{c_{(n:n)i}}\right] \quad ,\underline{\gamma}^{*'}C^{*-1}\underline{\gamma}^{*}=\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}^{2}}{c_{(n:n)i}}\right],$$

$$\underline{\gamma}^{*'}C^{*-1}\underline{1}=\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right],\ \underline{1}'C^{*-1}\underline{Z}^{*}=\left[\sum_{i=1}^{m+1}\frac{Z_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+1}^{n}\frac{Z_{(n:n)i}}{c_{(n:n)i}}\right]$$

and

$$\underline{\gamma}^{*'}C^{*-1}\underline{Z}^{*}=\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}Z_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}Z_{(n:n)i}}{c_{(n:n)i}}\right].$$

Now using the above results, (16) and (17) reduces to

$$\tilde{\psi}=\frac{d\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}Z_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}Z_{(n:n)i}}{c_{(n:n)i}}\right]+\left[\sum_{i=1}^{m+1}\frac{Z_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{Z_{(n:n)i}}{c_{(n:n)i}}\right]}{\left[d^{2}\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}^{2}}{c_{(n:n)i}}\right]+2d\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right]+\left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{1}{c_{(n:n)i}}\right]}$$

and

$$Var(\tilde{\psi})=\frac{d^{2}\psi^{2}}{\left[d^{2}\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}^{2}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}^{2}}{c_{(n:n)i}}\right]+2d\left[\sum_{i=1}^{m+1}\frac{\gamma_{(1:n)i}}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{\gamma_{(n:n)i}}{c_{(n:n)i}}\right]+\left[\sum_{i=1}^{m+1}\frac{1}{c_{(1:n)i}}+\sum_{i=m+2}^{n}\frac{1}{c_{(n:n)i}}\right]}$$

Thus the theorem is proved.

3. Estimation of the Mean of the Normal Distribution with Known Coefficient of Variation by Extreme Ranked Set Sampling

A continuous random variable Z is said to have normal distribution with location parameter ψ and scale parameter $d\psi$, if its pdf is given by

$$h(z;\psi,d\psi)=\frac{1}{d\psi\sqrt{2\pi}}\exp\left\{-\frac{(z-\psi)^{2}}{2d^{2}\psi^{2}}\right\},z\in R,\psi>0\,,d>0. \tag{19}$$

We will write $N(\psi, d\psi)$ to denote normal distribution defined in (19). The mean and variance of this distribution are given by $E(Z) = \psi$ and $Var(Z) = d^2\psi^2$, where d is the known coefficient of variation.

We have evaluated the coefficient of $Z_{(i:n)j}, i = 1, n; j = 1,2,...,n$ in the BLUE $\tilde{\psi}$ of ψ defined in (5) and (13)(for even and odd values of n) and variance of $\tilde{\psi}$ defined in (6) and (14) (for even and odd cases of n), d=0.25, 0.50 and n=2(1)10 are given in Table 3.1.Also we have evaluated the relative efficiency $RE_1 = \frac{Var(\hat{\psi})}{Var(\tilde{\psi})}$ of $\tilde{\psi}$ defined in (5) and (13) (for even and odd values of n) related to $\hat{\psi}$ defined in (3) and (11) (for even and odd values of n)are given in Table 3.1. From Table 3.1 it may be noted that in all the cases efficiency of our estimator $\tilde{\psi}$ is much better than that of $\hat{\psi}$.

4. Estimation of the Mean of the Logistic Distribution with Known Coefficient of Variation by Extreme Ranked Set Sampling

A continuous random variable Z is said to have logistic distribution with location parameter ψ and scale parameter $d\psi$, if its pdf is given by

$$h(z;\psi, d\psi) = \frac{\pi}{\sqrt{3}} \frac{\exp\left\{-\frac{\pi}{\sqrt{3}}\left(\frac{z-\psi}{d\psi}\right)\right\}}{c\mu\left[1+\exp\left\{-\frac{\pi}{\sqrt{3}}\left(\frac{z-\psi}{d\psi}\right)\right\}\right]^2}, z \in R, \psi > 0, d > 0. \qquad (20)$$

We will write $LD(\psi, d\psi)$ to denote the logistic distribution defined in (20) .The logistic distribution is a well-known and widely used statistical distribution because of its historical importance and its simplicity as growth curve (see, Erkelens (1968)). The mean and variance of this distribution given in (20) are $E(Z) = \psi$ and $Var(Z) = d^2\psi^2$, where d is the known coefficient of variation. We have evaluated the coefficient of $Z_{(i:n)j}, i = 1, n; j = 1,2,...,n$ in the BLUE $\tilde{\psi}$ of ψ defined in (5) and (13) (for even and odd values of n) and variance of $\tilde{\psi}$ defined in (6) and (14) (for even and odd cases of n), for d=0.15 and 0.20 and n=2(1)10 are given in Table 4.1. Also we have evaluated the relative efficiency $RE_1 = \frac{Var(\hat{\psi})}{Var(\tilde{\psi})}$ of $\tilde{\psi}$ defined in (5) and (13) (for even and odd values of n) related to $\hat{\psi}$ defined in (2) and (for even and odd values of n) are given in Table 4.1.From Table 4.1 it may be noted that all the cases efficiency of our estimator $\tilde{\psi}$ is much better than that of $\hat{\psi}$.

5. Estimation of the Mean of the Double Exponential Distribution with Known Coefficient of Variation by Extreme Ranked Set Sampling

A continuous random variable Z is said to have double exponential distribution with location parameter ψ and scale parameter $d\psi$, if its pdf is given by

$$h(z;\psi,d\psi) = \frac{1}{d\psi\sqrt{2}} \exp\left\{-\sqrt{2}\left|\frac{z-\psi}{d\psi}\right|\right\}, z \in R, \psi > 0, d > 0. \tag{21}$$

We will write DE ($\psi, d\psi$) to denote the double exponential distribution defined in (21). The mean and variance of this distribution given in (21) are $E(Z) = \psi$ and $Var(Z) = d^2\psi^2$, where d is the known coefficient of variation. We have evaluated the coefficient of $Z_{(i:n)j}, i = 1, n; j = 1,2,...,n$ in the BLUE $\tilde{\psi}$ of ψ defined in (5) and (13) (for even and odd values of n) and variance of $\tilde{\psi}$ defined in (6) and (14) (for even and odd cases of n), d=0.15,0.20 and n=2(1)10 are given in Table 5.1. Also we have evaluated the relative efficiency $RE_1 = \frac{Var(\hat{\psi})}{Var(\tilde{\psi})}$ of $\tilde{\psi}$ defined in (5) and (13) (for even and odd values of n) related to $\hat{\psi}$ defined in (3) and (11) (for even and odd cases of n) are given in Table 5.1.From table 5.1 it may be noted that all the cases efficiency of our estimator $\tilde{\psi}$ is much better than that of $\hat{\psi}$.

6. Real Life Example

For the years 1956 to 1974, Robert (1979) has provided the one-hour average sulphur dioxide concentration (in pphm) from Long Beach, California. From this data we observe the data for the months of June and July and are respectively (3,23, 13,6, 13, 10, 12, 7, 9, 10, 30,7,19,10, 12, 15,13,16,14) and (14,18,37, 8,14,8,10,4,16,18, 13,8,22, 13, 25,20,23, 25,9). Clearly for the two data sets the coefficient of variations are respectively 0.47 and 0.49. Also using Shapiro-Wilk normality test and using R programme the p values for the data sets are respectively 0.10 and 0.28. Hence the above two data sets follows normal distribution at 5% level of significant. Now clearly the coefficient variations for the data sets of the months of June and July is approximately 0.5. Now by knowing the coefficient of variation 0.5, consider the data set for the month of September. The data set for the month of September is (33,13,32,17,13, 12,11,15,4,14,22,10,26,33,25,38,21,11,25). Clearly the coefficient of variation for the month of September data is 0.47, and is approximately 0.5. Also by Shapiro-wilk normality test, the p value of the data set for the month of September is 0.27. Hence the data set for the month of September follows normal distribution at 5% level of significant.

Now we take four sets of random samples of sizes four each from the data of the month of September. The four data sets taken at random are (38,25,21,25), (5,11,13,13),(33,10,22,14) and (33,4,12,11). Now the extreme ranked set sample from the above four data sets is (21,5,33,33). Now using Table 3.1, for n=4 and d=0.5 the BLUE of the population mean μ of the normal data set for the month of September using ERSS is

$$\tilde{\psi} = 0.09592*21+0.09592*5+0.29937*33+0.29937*33=22.5$$

and $Var(\tilde{\psi}) = 0.02430\,\psi^2$.

7. Conclusion

Using ERSS one can estimate the BLUE of the location parameter of a location scale family of distributions when the location parameter is proportional to the scale parameter. Also it may be noted that in all the cases calculated in this work, the BLUE of the mean of normal, logistic and double exponential distributions with known coefficient of variation using ERSS is much better than the competing estimator using ERSS considered for comparison.

Acknowledgments

The authors are highly thankful for the valuable comments of the referee.

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Table 3.1: Coefficients of $Z_{(i:n)j}$ in the BLUE, $\tilde{\psi}$ $V_1 = \frac{Var(\tilde{\psi})}{\psi^2}$, $V_2 = \frac{Var(\hat{\psi})}{\psi^2}$ and $RE_1 = \frac{V_2}{V_1}$, the relative efficiency of $\tilde{\psi}$ relative to $\hat{\psi}$ for different values of d=0.25, 0.5.

	d	Coefficients										V_1	V_2	RE_1
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}			
2	0.25	0.42110	0.55939									0.02089	0.02130	1.01963
	0.50	0.33249	0.59379									0.07893	0.08521	1.07956
3	0.25	0.29081	0.29081	0.44689								0.01290	0.01311	1.01628
	0.50	0.21438	0.21438	0.52888								0.05198	0.05245	1.00904
4	0.25	0.17413	0.17413	0.29481	0.29481							0.01583	0.01721	1.23671
	0.50	0.09592	0.09592	0.29937	0.29937							0.02430	0.03073	1.26461
5	0.25	0.14651	0.14651	0.14651	0.26662	0.26662						0.00578	0.00583	1.00865
	0.50	0.07571	0.07571	0.07571	0.28610	0.28610						0.02024	0.02331	1.15168
6	0.25	0.10348	0.10348	0.10348	0.19945	0.19945	0.19945					0.00394	0.00433	1.09898
	0.50	0.04357	0.04357	0.04357	0.04357	0.04357	0.04357					0.01165	0.01633	1.40172
7	0.25	0.09292	0.09292	0.09292	0.09292	0.18783	0.18783	0.18783				0.00259	0.00291	1.12355
	0.50	0.03661	0.03661	0.03661	0.03661	0.18844	0.18844	0.18844				0.01107	0.01429	1.29088
8	0.25	0.07146	0.07146	0.07146	0.07146	0.15043	0.15043	0.15043	0.15043			0.00259	0.00291	1.12355
	0.50	0.02391	0.02391	0.02391	0.02391	0.14202	0.14202	0.14202	0.14202			0.00773	0.01165	1.50712
9	0.25	0.06620	0.06620	0.06620	0.06620	0.06620	0.14437	0.14437	0.14437	0.14437		0.00235	0.00251	1.06809
	0.50	0.02064	0.02064	0.02064	0.02064	0.02064	0.13966	0.13966	0.13966	0.13966		0.00716	0.01005	1.40363
10	0.25	0.05360	0.05360	0.05360	0.05360	0.05360	0.12062	0.12062	0.12062	0.12062	0.12062	0.00187	0.00215	1.14973
	0.50	0.07303	0.07303	0.07303	0.07303	0.07303	0.11686	0.11686	0.11686	0.11686	0.11686	0.00074	0.00077	1.04054

Table 4.1: Coefficients of $Z_{(i:n)j}$ in the BLUE $\tilde{\psi}$, $V_1 = \frac{Var(\tilde{\psi})}{\psi^2}$, $V_2 = \frac{Var(\hat{\psi})}{\psi^2}$ and $RE_1 = \frac{V_2}{V_1}$, the relative efficiency of $\tilde{\psi}$ relative to $\hat{\psi}$ for different values of d=0.15, 0.20.

n	D	Coefficients										V_1	V_2	RE_1
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}			
2	0.15	0.45553	0.53767									0.00778	0.00783	1.00643
	0.20	0.43952	0.54846									0.01375	0.01392	1.01236
3	0.15	0.31306	0.31306	0.40172								0.00499	0.00523	1.04810
	0.20	0.30335	0.30335	0.42359								0.00901	0.00930	1.03219
4	0.15	0.20733	0.20733	0.28143	0.28143							0.00322	0.00330	1.02484
	0.20	0.19163	0.19163	0.28874	0.28874							0.00563	0.00586	1.04085
5	0.15	0.17230	0.17230	0.17230	0.24403	0.24403						0.00266	0.00266	1.00000
	0.20	0.16033	0.16033	0.16033	0.25596	0.25596						0.00472	0.00473	1.00212
6	0.15	0.13054	0.13054	0.13054	0.19132	0.19132	0.19132					0.00201	0.00208	1.03483
	0.20	0.11727	0.11727	0.11727	0.19619	0.19619	0.19619					0.00348	0.00370	1.06322
7	0.15	0.11586	0.11586	0.11586	0.11586	0.17474	0.17474	0.17474				0.00179	0.00179	1.00000
	0.20	0.10470	0.10470	0.10470	0.10470	0.18222	0.18222	0.18222				0.00314	0.00319	1.01592
8	0.15	0.09388	0.09388	0.09388	0.09388	0.14513	0.14513	0.14513	0.14513			0.00145	0.00152	1.04828
	0.20	0.08252	0.08252	0.08252	0.08252	0.14859	0.14859	0.14859	0.14859			0.00250	0.00270	1.08000
9	0.15	0.08609	0.08609	0.08609	0.08609	0.08609	0.13601	0.13601	0.13601	0.13601		0.00134	0.00136	1.01493
	0.20	0.07605	0.07605	0.07605	0.07605	0.07605	0.14113	0.14113	0.14113	0.14113		0.00233	0.00241	1.03433
10	0.15	0.07263	0.07263	0.07263	0.07263	0.07263	0.11699	0.11699	0.11699	0.11699	0.11699	0.00113	0.00120	1.06195
	0.20	0.06270	0.06270	0.06270	0.06270	0.06270	0.11956	0.11956	0.11956	0.11956	0.11956	0.00194	0.00213	1.09794

Table 5.1: Coefficients of $Z_{(i:n)j}$ in the BLUE $\tilde{\psi}$, $V_1 = \frac{Var(\tilde{\psi})}{\psi^2}$, $V_2 = \frac{Var(\hat{\psi})}{\psi^2}$ and $RE_1 = \frac{V_2}{V_1}$,the relative efficiency of $\tilde{\psi}$ relative to $\hat{\psi}$ for different values of d=0.15,0.20.

n	d	Coefficients										V_1	V_2	RE_1
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}			
2	0.15	0.45733	0.53638									0.00803	0.00809	1.00747
	0.20	0.44200	0.54688									0.01421	0.01437	1.01126
3	0.15	0.31407	0.31407	0.39918								0.01075	0.01131	1.05209
	0.20	0.30492	0.30492	0.42031								0.01944	0.02010	1.03395
4	0.15	0.20876	0.20876	0.28068	0.28068							0.00925	0.00945	1.02162
	0.20	0.19359	0.19359	0.28793	0.28793							0.01618	0.01681	1.03894
5	0.15	0.17305	0.17305	0.17305	0.24318	0.24318						0.00935	0.00936	1.00107
	0.20	0.16143	0.16143	0.16143	0.25497	0.25497						0.01663	0.01664	1.00060
6	0.15	0.13107	0.13107	0.13107	0.19108	0.19108	0.19108					0.00450	0.00466	1.03556
	0.20	0.11799	0.11799	0.11799	0.19597	0.19597	0.19597					0.00780	0.00828	1.06154
7	0.15	0.11605	0.11605	0.11605	0.11605	0.17460	0.17460	0.17460				0.00837	0.00840	1.00358
	0.20	0.10496	0.10496	0.10496	0.10496	0.18207	0.18207	0.18207				0.01470	0.01484	1.01633
8	0.15	0.09383	0.09383	0.09383	0.09383	0.14515	0.14515	0.14515	0.14515			0.00756	0.00791	1.04630
	0.20	0.08245	0.08245	0.08245	0.08245	0.14861	0.14861	0.14861	0.14861			0.01300	0.01407	1.08231
9	0.15	0.08586	0.08586	0.08586	0.08586	0.08586	0.13615	0.13615	0.13615	0.13615		0.00762	0.00772	1.01312
	0.20	0.07572	0.07572	0.07572	0.07572	0.07572	0.14127	0.14127	0.14127	0.14127		0.01324	0.01373	1.03701
10	0.15	0.07222	0.07222	0.07222	0.07222	0.07222	0.11713	0.11713	0.11713	0.11713	0.11713	0.00697	0.00737	1.05739
	0.20	0.06217	0.06217	0.06217	0.06217	0.06217	0.11966	0.11966	0.11966	0.11966	0.11966	0.01191	0.01310	1.09992