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Inference on $P(X < Y)$ Based on Records for Bivariate Normal Distribution

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ABSTRACT

In this paper, we consider the problem of estimation of $R = P(X < Y)$, when (X, Y) follows bivariate normal distribution. The Maximum likelihood estimates and Bayes estimates (BEs) of R are obtained based on record values and its concomitants. BEs are obtained based on both symmetric and asymmetric loss functions. The bootstrap and credible confidence intervals for R are also obtained. Monte Carlo simulations are carried out to study the accuracy of the proposed estimators. A real data set is also used to illustrate the inferential procedures developed in this paper.

1. Introduction

In the context of reliability the stress-strength model describes the life of a component which has a random strength Y and is subjected to a random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X < Y$. Thus $R = P(X < Y)$ is a measure of component reliability. Nowadays the stress-strength relationship is studied in many branches of sciences and social sciences such as psychology, medicine, pedagogy, pharmaceutics and engineering. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels. Some examples are as follows (see, Nadarajah and Kotz, 2006). If X represents the maximum chamber pressure generated by ignition of a solid propellant and Y represents the strength of the rocket chamber, then R is the successful firing of the engine. Let Y and X be the remission times of two chemicals when they are administered to two kinds of mechanical systems, then inferences about R present a comparison of the

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effectiveness of the two chemicals. If X and Y are future observations on the stability of an engineering design, then R would be the predictive probability that X is less than Y. Similarly, if X and Y represent lifetimes of two electronic devices, then R is the probability that one fails before the other.

The estimation of R has been extensively investigated in the literature when X and Y are independent random variables belonging to the same family of distributions. However, there is a relative little work when X and Y are dependent random variables. The problem of estimating R, when the stress and strength are dependent, was considered by Abu-Salih and Shamseldin (1988), Samawi *et al.* (2016), Awad *et al.* (1981), Jana and Roy (1994) and Cramer (2001). Estimation of R when (X, Y) follows bivariate normal has been discussed by Enis and Geisser (1971) and Mukherjee and Saran (1985). Jana(1994) and Hanagal (1995) discussed the estimation procedure of R when (X, Y) follows Marshall-Olkin bivariate exponential distribution. Hanagal (1997) discussed the estimation of R when (X, Y) has a bivariate Pareto distribution. Nguimkeu *et al.* (2014) considered interval estimation of stress-strength reliability with bivariate normal variables. Recent advances on Bayesian inference for $P(X < Y)$ have been discussed by Ventura and Recugno (2011).

The applications of $R = P(X < Y)$ are not limited to reliability and engineering, it has lot of applications in medicine, psychology, environmental studies etc. For example in medical studies, if X and Y represent the outcome of control and experimental treatments, then R can be interpreted as the effectiveness of the treatment. In the study of water quality in freshwater, if Y represents the concentration of dissolved trace metals such as zinc, copper or lead in water and X represents the corresponding worldwide water quality standards of that metal in water, then $P(Y < X) = 1 - R$ can be considered as the probability that the metal concentration in freshwater is lower than the corresponding worldwide standard.

Record value data arise in a wide variety of practical situations. Examples include destructive stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events and oil and mining surveys. Interest in records has increased steadily over the years since Chandler (1952) formulation. Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called an upper record if $X_i < X_j$ for every $i < j$. In an analogous way, we can define lower record values. Useful surveys on record values are given in Ahsanullah (1995) and Arnold *et al.* (1998). Let (X_1, Y_1), (X_2, Y_2),... be a sequence of iid bivariate random variables with common continuous joint cdf $F(x, y)$, $(x, y) \in R \times R$. Let $F_X(x)$ and $F_Y(y)$ be the marginal cdfs of X and Y respectively. Let $R_n, n \geq 1$ be the sequence of upper record values arising from the sequence of X's. Then the Y-variate associated with the X-value, which is qualified as the nth record will be called the concomitant of the nth record and will be denoted by $R_{[n]}$. For a detailed discussion on the distribution theory of record values and concomitants of record values see, Arnold *et al.* (1998), Ahsanullah and Nevzorov (2000), Ahsanullah and Shakil (2013), Khan and Arshad (2016) and Arshad and Jamal (2019). Chacko and Thomas (2006, 2008) considered the problem of estimation of parameters of Morgenstern type bivariate logistic distribution and bivariate normal distribution based on concomitants of record values. The joint pdf of first n upper record values and its concomitants $(\mathbf{R}_{(\mathbf{n})}, \mathbf{R}_{[\mathbf{n}]}) = ((R_{(1)}, R_{[1]}),\ (R_{(2)}, R_{[2]})), \dots, (R_{(n)}, R_{[n]}))$ is given by

$$f_{(\mathbf{R}_{(\mathbf{n})},\mathbf{R}_{[\mathbf{n}]})}\left(\mathbf{r}_{(\mathbf{n})}, \mathbf{r}_{[\mathbf{n}]}\right) = f\left(r_{[i]} \middle| r_{(i)}\right) f_{1,2,\dots,n}\left(r_{(1)}, r_{(2)}, \dots, r_{(n)}\right), \qquad (1.1)$$

where $f_{1,2,\dots,n}(r_{(1)}, r_{(2)}, \dots, r_{(n)})$ is the joint pdf of first n upper record values and is given by

$$f_{1,2,\dots,n}(r_{(1)}, r_{(2)}, \dots, r_{(n)}) = f(r_{(n)}) \prod_{i=1}^{n-1} \frac{f(r_{(i)})}{1 - F(r_{(i)})}.$$

Suppose in an experiment, individuals are measured based on an inexpensive test and only those individuals whose measurement breaks the previous records are retained for the measurement based on an expensive test, then the resulting data involves record values and concomitants of record values.

In this paper, we focus on estimation of $R = P(X < Y)$ based on record values and its concomitants, when (X, Y) follows bivariate normal distribution (BVND). A random variable (X, Y) follows BVND if its pdf is given by

$$f(x, y) = \begin{cases} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}, \\ (x, y) \in R \times R; -\infty < \mu_1, \mu_2 < \infty; \sigma_1, \sigma_2 > 0; |\rho| < 1. \\ 0, \text{otherwise} \end{cases} \qquad (1.2)$$

If (X, Y) follows BVND with pdf defined in (1.2), then R is given by

$$R = P(X < Y)$$
$$= \Phi\left[\frac{\mu_2-\mu_1}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\right], \tag{1.3}$$

where Φ is the cdf of standard normal distribution. If we denote $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ as the vector of parameters then we can write

$$R = R(\theta)$$
$$= \Phi\left[\frac{\mu_2-\mu_1}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\right].$$

In this paper, first we consider the maximum likelihood estimation of R for BVND based on record values and its concomitants. The maximum likelihood method is the most widely used estimation method. However, in many practical situations, occurrences of record values are very rare and sample sizes are often very small. Thereby maximum likelihood estimator (MLE) may produce substantial bias and also intervals based on the asymptotic normality of MLEs do not perform well. Hence in this paper we also consider Bayesian estimation of R for BVND based on record values and its concomitants.

The organization of the paper is as follows. In section 2, we consider the maximum likelihood estimation of R using record values and its concomitants. The bootstrap confidence intervals based on MLE are also included in this section. In section 3, we consider the Bayesian estimation of R using importance sampling method. Section 4 is devoted to some simulation studies. In section 5 a real data on water quality is considered for illustration and finally in section 6 we give some concluding remarks.

2. Maximum Likelihood Estimation

In this section, we obtain the MLE of R for BVND given in (1.2) using record values and its concomitants. Let $(R_{(i)}, R_{[i]}), i = 1,2, \dots, n$ be the first n upper record values and its concomitants arising from BVND with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. Then from (1.1) the likelihood function is given by

$$L(\mu_1, \mu_2, \sigma_1\sigma_2, \rho) = \prod_{i=1}^{n} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} exp\frac{-1}{2(1-\rho^2)}\{(\frac{r_{(i)}-\mu_1}{\sigma_1})^2$$
$$-2\rho(\frac{r_{(i)}-\mu_1}{\sigma_1})(\frac{r_{[i]}-\mu_2}{\sigma_2}) + (\frac{r_{[i]}-\mu_2}{\sigma_2})^2\}$$
$$\prod_{i=1}^{n-1} \frac{1}{\left(1-\Phi(\frac{r_{(i)}-\mu_1}{\sigma_1})\right)}. \tag{2.1}$$

Then the log-likelihood function is given by

$$logL(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = -n\log 2\pi - n\log\sigma_1 - n\log\sigma_2 - \frac{n}{2}\log(1-\rho^2)$$

$$-\frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\{(\frac{r_{(i)}-\mu_1}{\sigma_1})^2+(\frac{r_{[i]}-\mu_2}{\sigma_2})^2$$

$$-2\rho(\frac{r_{(i)}-\mu_1}{\sigma_1})(\frac{r_{[i]}-\mu_2}{\sigma_2})\}-\sum_{i=1}^{n-1}\log(1-\Phi(\frac{r_{(i)}-\mu_1}{\sigma_1})).$$

For notation convenience, we denote $A_i=\frac{r_{(i)}-\mu_1}{\sigma_1}$ and $B_i=\frac{r_{[i]}-\mu_2}{\sigma_2}$.

Thus we have

$$\frac{\partial \log L}{\partial \mu_1}=\sum_{i=1}^{n}\frac{1}{\sigma_1(1-\rho^2)}[A_i-\rho B_i]-\sum_{i=1}^{n-1}\frac{\phi(A_i)}{\sigma_1[1-\Phi(A_i)]},$$

$$\frac{\partial \log L}{\partial \mu_2}=\sum_{i=1}^{n}\frac{1}{\sigma_2(1-\rho^2)}[B_i-\rho A_i]$$

$$\frac{\partial \log L}{\partial \sigma_1}=\frac{-n}{\sigma_1}+\sum_{i=1}^{n}\frac{1}{\sigma_1(1-\rho^2)}[(A_i)^2-\rho A_iB_i]-\sum_{i=1}^{n-1}\frac{\phi(A_i)}{\sigma_1[1-\Phi(A_i)]}A_i,$$

$$\frac{\partial \log L}{\partial \sigma_2}=\frac{-n}{\sigma_2}+\sum_{i=1}^{n}\frac{1}{\sigma_2(1-\rho^2)}[(B_i)^2-\rho A_iB_i]$$

$$\frac{\partial \log L}{\partial \rho}=\frac{n\rho}{1-\rho^2}-\frac{1}{(1-\rho^2)^2}\sum_{i=1}^{n}[\rho(A_i)^2+\rho(B_i)^2-(1+\rho^2)A_iB_i].$$

Then the normal equatios are

$$\sum_{i=1}^{n}\frac{1}{\sigma_1(1-\rho^2)}[A_i-\rho B_i]-\sum_{i=1}^{n-1}\frac{\phi(A_i)}{\sigma_1[1-\Phi(A_i)]}=0,$$

$$\sum_{i=1}^{n}\frac{1}{\sigma_2(1-\rho^2)}[B_i-\rho A_i]=0,$$

$$\frac{-n}{\sigma_1}+\sum_{i=1}^{n}\frac{1}{\sigma_1(1-\rho^2)}[(A_i)^2-\rho A_iB_i]-\sum_{i=1}^{n-1}\frac{\phi(A_i)}{\sigma_1[1-\Phi(A_i)]}A_i=0,$$

$$\frac{-n}{\sigma_2}+\sum_{i=1}^{n}\frac{1}{\sigma_2(1-\rho^2)}[(B_i)^2-\rho A_iB_i]=0$$

and

$$\frac{n\rho}{1-\rho^2}-\frac{1}{(1-\rho^2)^2}\sum_{i=1}^{n}[\rho(A_i)^2+\rho(B_i)^2-(1+\rho^2)A_iB_i]=0.$$

The MLEs of μ_1, μ_2, σ_1,σ_2 and ρ can be obtained by solving the above non-linear equations using the Newton-Raphson method or any other numerical methods. Let $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\rho}$ be the MLEs of μ_1, μ_2, σ_1, σ_2 and ρ obtained by solving the non-linear equations, then the MLE of R is given by

$$\hat{R}_{ML}=\Phi\left[\frac{\hat{\mu}_2-\hat{\mu}_1}{\sqrt{\hat{\sigma}_1^2+\hat{\sigma}_2^2-2\hat{\rho}\hat{\sigma}_1\hat{\sigma}_2}}\right]. \tag{2.2}$$

***Remark 2.1* :** In many practical situations, occurrences of record values are very rare and sample sizes are often very small. Thereby intervals based on the asymptotic normality of MLEs do not perform well. Hence in the next subsection we obtain bootstrap confidence intervals for R.

2.1 Bootstrap Confidence Intervals

In this subsection, we consider bootstrap confidence intervals (CIs) for R based on MLEs. We consider both percentile and bias-corrected & accelerated (BCa) confidence intervals for R. For more details on bootstrap confidence intervals see, Carpenter and Bithell (2000). The algorithm for bootstrap CIs is the following;

1. Compute the MLEs $\hat{\mu}_1{}^{(0)}$, $\hat{\mu}_2{}^{(0)}$,$\hat{\sigma}_1{}^{(0)}$,$\hat{\sigma}_2{}^{(0)}$,$\hat{\rho}^{(0)}$ of $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho$ using original observations and find the MLE $\hat{R}_{ML}$.
2. Generate a bootstrap sample using $\hat{\mu}_1{}^{(0)}$, $\hat{\mu}_2{}^{(0)}$, $\hat{\sigma}_1{}^{(0)}$, $\hat{\sigma}_2{}^{(0)}$, $\hat{\rho}^{(0)}$ and obtain the MLEs $\hat{\mu}_1{}^{(k)}$, $\hat{\mu}_2{}^{(k)}$, $\hat{\sigma}_1{}^{(k)}$, $\hat{\sigma}_2{}^{(k)}$, $\hat{\rho}^{(k)}$ using the bootstrap sample.
3. Obtain the MLE of $\hat{R}_k = R(\hat{\mu}_1{}^{(k)}, \hat{\mu}_2{}^{(k)}, \hat{\sigma}_1{}^{(k)}, \hat{\sigma}_2{}^{(k)}, \hat{\rho}^{(k)})$.
4. put $k = k + 1$.
5. Repeat steps (2) to (4) B times to have $\hat{R}_k$ for $k = 1,2, \dots, B$
6. Arrange $\hat{R}_k$ for $k = 1,2, \dots, B$ in ascending order as $\hat{R}_{(1)} \le \hat{R}_{(2)}, \dots, \le \hat{R}_{(B)}$. Then

(a) The $100(1-\alpha)$ percentile bootstrap CI for R is given by

$$\left(\hat{R}_{([B(\alpha/2)])}, \hat{R}_{([B(1-\alpha/2)])}\right).$$

(b) The $100(1-\alpha)$ bootstrap BCa CI for R is given by

$(\hat{R}_{([B\alpha_1])}, \hat{R}_{([B\alpha_2])})$, where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1 - \hat{a}(\hat{z}_0 + z_{\alpha/2})}\right), \alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{1-\alpha/2}}{1 - \hat{a}(\hat{z}_0 + z_{1-\alpha/2})}\right)$$

and

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\{\hat{R}_k < \hat{R}_{ML}\}}{B}\right), \hat{a} = \frac{\sum_{i=1}^{n} (\hat{R}^{(.)} - \hat{R}^{(i)})^3}{6\{(\hat{R}^{(.)} - \hat{R}^{(i)})^2\}^{\frac{3}{2}}}$$

with $\hat{R}^{(.)} = \sum_{i=1}^{n} \frac{\hat{R}^{(i)}}{n}$ and $\hat{R}^{(i)}$ being the estimate of R calculated on the original sample with the ith couple of observations deleted.

3. Bayesian Estimation

In this section, we consider Bayesian estimation of R for BVND under symmetric as well as asymmetric loss functions. A symmetric loss function is the squared error loss (SEL) function which is defined as

$$L_1\left(d(\mu), \hat{d}(\mu)\right) = \left(\hat{d}(\mu) - d(\mu)\right)^2, \tag{3.1}$$

where $\hat{d}(\mu)$ is an estimate of $d(\mu)$. The Bayes estimate of μ under L_1 is the posterior mean of μ. An asymmetric loss function is the LINEX loss (LL) function which is defined as

$$L_2\left(d(\mu),\hat{d}(\mu)\right) = e^{h(\hat{d}(\mu)-d(\mu))} - h\left(\hat{d}(\mu) - d(\mu)\right) - 1, h \neq 0. \qquad (3.2)$$

The Bayes estimate of $d(\mu)$ for the loss function L_2 can be obtained as

$$\hat{d}_{LB} = \frac{-1}{h}\log\left\{E_\mu\left(e^{-h\mu}\middle|\underline{x}\right)\right\}, \qquad (3.3)$$

provided E_μ exists.

Let $(R_{(i)}, R_{[i]}), i = 1,2, \dots, n$ be the vector of record values and its concomitants arising from BVND$(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$. Then from (2.1) the likelihood function is given by

$$L(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho) = \prod_{i=1}^{n} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} exp\frac{-1}{2(1-\rho^2)}\{(\frac{r_{(i)}-\mu_1}{\sigma_1})^2$$

$$-2\rho\left(\frac{r_{(i)}-\mu_1}{\sigma_1}\right)\left(\frac{r_{[i]}-\mu_2}{\sigma_2}\right) + \left(\frac{r_{[i]}-\mu_2}{\sigma_2})^2\right\}$$

$$\prod_{i=1}^{n-1} \frac{1}{(1-\Phi(\frac{r_{(i)}-\mu_1}{\sigma_1}))}. \qquad (3.4)$$

Assume that the prior distributions of $\mu_1|\sigma_1 \sim N(\mu_{01},\sigma_1^2)$, $\mu_2|\sigma_2 \sim N(\mu_{02},\sigma_2^2)$, $\sigma_1^2 \sim$ Inverse Gamma$(b, a/2)$,$\sigma_2^2 \sim$ Inverse Gamma$(d, c/2)$ and $\rho \sim u(-1,1)$. Therefore the prior density functions of $\mu_1,\mu_2,\sigma_1,\sigma_2$ and ρ are respectively given by

$$\pi_1(\mu_1|\sigma_1) = \frac{1}{\sqrt{2\pi}\sigma_1} exp\frac{-1}{2\sigma_1^2}(\mu_1-\mu_{01})^2, \qquad (3.5)$$

$$\pi_2(\mu_2|\sigma_2) = \frac{1}{\sqrt{2\pi}\sigma_2} exp\frac{-1}{2\sigma_2^2}(\mu_2-\mu_{02})^2 \qquad (3.6)$$

$$\pi_3(\sigma_1) = \frac{a^b}{\Gamma(b)2^{b-1}}\sigma_1^{-2b-1} exp\frac{-a}{2\sigma_1^2}, \qquad (3.7)$$

$$\pi_4(\sigma_2) = \frac{c^d}{\Gamma(d)2^{d-1}}\sigma_2^{-2d-1} exp\frac{-c}{2\sigma_2^2} \qquad (3.8)$$

and

$$\pi_5(\rho) = \frac{1}{2}; -1 < \rho < 1. \qquad (3.9)$$

Thus the joint prior distribution of $\mu_1,\mu_2,\sigma_1,\sigma_2$ and ρ is given by

$$\pi(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho) = \frac{1}{2\pi\sigma_1\sigma_2}\frac{a^b}{\Gamma(b)2^{b-1}}\frac{c^d}{\Gamma(d)2^{d-1}}\sigma_1^{-2b-1}\sigma_2^{-2d-1}$$

$$exp\frac{-1}{2}\{(\frac{\mu_1-\mu_{01}}{\sigma_1})^2 + (\frac{\mu_2-\mu_{02}}{\sigma_2})^2 + \frac{a}{\sigma_1^2} + \frac{c}{\sigma_2^2}\}.$$

Then the joint posterior density of θ=(μ_1,μ_2, σ_1, σ_2,ρ) given the data is

$$\pi^*(\theta|data) = \frac{L(\theta)\pi(\theta)}{\int_\theta L(\theta)\pi(\theta)d\theta}. \tag{3.10}$$

Therefore the Bayes estimate of $R(\theta)$ under SEL and LL are respectively given by

$$\widehat{R}_S(\theta) = \frac{\int_\theta R(\theta)L(\theta)\pi(\theta)d\theta}{\int_\theta L(\theta)\pi(\theta)d\theta}, \tag{3.11}$$

and

$$\widehat{R}_L(\theta) = \frac{-1}{h}\log\frac{\int_\theta e^{-hR(\theta)}L(\theta)\pi(\theta)d\theta}{\int_\theta L(\theta)\pi(\theta)d\theta}. \tag{3.12}$$

It is not possible to compute (3.11) and (3.12) explicitly. Thus we propose importance sampling method to find the Bayes estimates for R.

3.1 Importance Sampling Method

In this subsection, we consider the sequential importance sampling method to generate samples from the posterior distribution and then find the Bayes estimate of R (see, Tokdar and Kass, 2010). The numerator in the posterior distribution given in (3.10) can be written as

$$L(\theta)\pi(\theta) \propto Q(\theta)f_1(\mu_1|\sigma_1,\rho)f_2(\mu_2|\sigma_2,\rho)g_1(\sigma_1|\rho)g_2(\sigma_2|\rho)h(\rho),$$

where

$$Q(\theta) = \frac{1}{(n+(1-\rho^2))(1-\rho^2)^{\frac{n}{2}-1}} exp\,\frac{\rho}{1-\rho}\sum_{r=1}^{n}\left(\frac{r_{(i)}-\mu_1}{\sigma_1}\right)\left(\frac{r_{[i]}-\mu_2}{\sigma_2}\right)$$

$$\prod_{r=1}^{n-1}\frac{1}{\left[1-\Phi\left(\frac{r_{(i)}-\mu_1}{\sigma_1}\right)\right]}, \tag{3.13}$$

$$f_1(\mu_1|\sigma_1,\rho) = \frac{\sqrt{n+(1-\rho^2)}}{\sigma_1\sqrt{1-\rho^2}}\ exp\left\{\frac{-1}{2}\frac{n+(1-\rho^2)}{(1-\rho^2)\sigma_1^2}\Big(\mu_1 -\right.$$

$$nx+\mu01(1-\rho2)n+(1-\rho2)2, \tag{3.14}$$

$$f_2(\mu_2|\sigma_2,\rho) = \frac{\sqrt{n+(1-\rho^2)}}{\sigma_2\sqrt{1-\rho^2}} exp\left\{\frac{-1}{2}\frac{n+(1-\rho^2)}{(1-\rho^2)\sigma_2^2}\Big(\mu_2 -\right.$$

$$g_1(\sigma_1|\rho) = \sigma_1^{-2\left(\frac{n}{2}+b\right)-1}$$

$$exp\left\{\frac{-1}{2\sigma_1^2}\left(a+\frac{ns_1^2}{1-\rho^2}+\frac{n(\bar{x}-\mu_{01})^2}{n+(1-\rho^2)}\right)\right\}, \tag{3.16}$$

$$g_2(\sigma_2|\rho) = \sigma_2^{-2\left(\frac{n}{2}+d\right)-1}$$
$$exp\left\{\frac{-1}{2\sigma_2^2}\left(c + \frac{ns_2^2}{1-\rho^2} + \frac{n(\bar{y}-\mu_{02})^2}{n+(1-\rho^2)}\right)\right\} \quad (3.17)$$

and

$$h(\rho) = \frac{1}{2}. \quad (3.18)$$

Here $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} r_{(i)}$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} r_{[i]}$, $s_1 = \frac{1}{n}\sum_{i=1}^{n} (r_{(i)} - \bar{x})^2$ and $s_2 = \frac{1}{n}\sum_{i=1}^{n} (r_{[i]} - \bar{y})^2$. From (3.14) to (3.18) we can see that

$$\mu_1|\sigma_1, \rho \sim N\left(\frac{n\bar{x} + \mu_{01}(1-\rho^2)}{n + (1-\rho^2)}, \frac{\sigma_1^2(1-\rho^2)}{n + (1-\rho^2)}\right),$$

$$\mu_2|\sigma_2, \rho \sim N\left(\frac{n\bar{y} + \mu_{02}(1-\rho^2)}{n + (1-\rho^2)}, \frac{\sigma_2^2(1-\rho^2)}{n + (1-\rho^2)}\right),$$

$$\sigma_1^2|\rho \sim IG\left(\frac{n}{2} + b, a + \frac{ns_1^2}{1-\rho^2} + \frac{n(\bar{x}-\mu_{01})^2}{n + (1-\rho^2)}\right),$$

$$\sigma_2^2|\rho \sim IG\left(\frac{n}{2} + d, c + \frac{ns_2^2}{1-\rho^2} + \frac{n(\bar{y}-\mu_{02})^2}{n + (1-\rho^2)}\right)$$

and

$$\rho \sim U(-1,1).$$

Let $\theta^{(t)} = (\mu_1^{(t)}, \mu_2^{(t)}, \sigma_1^{(t)}, \sigma_2^{(t)}, \rho^{(t)})$, $t = 1,2, \dots, N$ be the observations generated from (3.14) to (3.18) respectively. Then by importance sampling method the Bayes estimators under SEL and LL given by (3.11) and (3.12) can be obtained as

$$\hat{R}_S = \frac{\sum_{t=1}^{N} R(\theta^{(t)})Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}, \quad (3.19)$$

and

$$\hat{R}_L = \frac{-1}{h}\log\left[\frac{\sum_{t=1}^{N} exp(-hR(\theta^{(t)})Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}\right]. \quad (3.20)$$

3.2 HPD Interval

In this subsection, we construct HPD interval under SEL for R as described in Chen and Shao (1999). Define $R_t = R(\theta^{(t)})$, where $\theta^{(t)}$ for $t = 1,2, \dots, N$ are posterior samples generated respectively from (3.14) to (3.18) for $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ. Define

$$w_t = \frac{Q(\theta^{(t)})}{\sum_{i=1}^{M} Q(\theta^{(t)})},$$

where $Q(\theta^{(t)})$ is given in (3.13). Let $R_{(t)}$ be the ordered values of R_t. Then the pth quantile of R can be estimated as

$$\hat{R}^{(p)} = \begin{cases} R_{(1)} \text{if} \quad p = 0 \\ R_{(i)} \quad if \quad \sum_{j=1}^{i-1} w_{(j)} < p < \sum_{j=1}^{i} w_{(j)}, \end{cases}$$

where $w_{(j)}$ is the weight function associated with jth ordered value $R_{(j)}$. Then the $100(1-\alpha)\%$, $0 < \alpha < 1$, confidence interval for R is given by $(\hat{R}^{(j/N)}, \hat{R}^{((j+[(1-\alpha)N])/N)})$, $j = 1,2,\dots,N-[(1-\alpha)N]$, where [.] is the greatest integer function. Then the required HPD interval for R is the interval with smallest width.

4. Simulation Study

In this section, we carry out a simulation study for illustrating the estimation procedures developed in previous sections. First we obtain the MLE of R using (2.2). We have obtained the bias and MSE of MLEs for different combinations of μ_1, μ_2 and ρ and are provided in table 1. The bootstrap CIs for R are also obtained. The average interval length (AIL) and coverage probability (CP) are also obtained and are included in table 1. For the simulation study for Bayesian estimation we took both informative and non-informative priors. We have considered two sets of informative priors, say Prior I and Prior II. The hyper parameters for Prior I and Prior II are given below.

Prior I : $\mu_{01} = 2$, $\mu_{02} = 2$, $a = 2.5$, $b = 2.25$,$c = 2.5$ and $d = 2.25$
Prior II : $\mu_{01} = 2$, $\mu_{02} = 2$, $a = 4$, $b = 3$,$c = 4$ and $d = 3$.

The values of a, b, c and d for Prior I and Prior II are chosen such that the means of σ_1^2 and σ_2^2 are fixed (equal to 1) and variances are high (Prior I) and low (Prior II). The non-informative prior is obtained by taking $\mu_{01} = 0$, $\mu_{02} = 0$, $a = 0$, $b = 0$,$c = 0$ and $d = 0$. We have obtained the Bayes estimator for R of BVND distribution under SEL and LL (with h=1) functions. The simulation studies were performed in R-program. For finding MLEs we used nlm function in R. For the simulation studies for Bayes estimators we use the following algorithm.

1. Generate n upper record values and its concomitants from BVND distribution with parameters $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$.
2. Calculate estimator of R using the generated values using importance sampling method as described below.

(a) Put $t = 1$

(b) Generate $\rho^{(t)}$ from $U(-1,1)$.

(c) Generate $(\sigma_1^{(t)})^2$ from $IG\left(\frac{n}{2}+b, a+\frac{ns_1^2}{1-(\rho^{(t)})^2}+\frac{n(\bar{x}-\mu_{01})^2}{n+(1-(\rho^{(t)})^2)}\right)$.

(d) Generate $(\sigma_2^{(t)})^2$ from $IG\left(\frac{n}{2}+d, c+\frac{ns_2^2}{1-(\rho^{(t)})^2}+\frac{n(\bar{y}-\mu_{02})^2}{n+(1-(\rho^{(t)})^2)}\right)$.

(e) Generate $\mu_1^{(t)}$ from $N\left(\frac{n\bar{x}+\mu_{01}(1-(\rho^{(t)})^2)}{n+(1-(\rho^{(t)})^2)}, \frac{(\sigma_1^{(t)})^2(1-(\rho^{(t)})^2)}{n+(1-(\rho^{(t)})^2)}\right)$.

(f) Generate $\mu_2^{(t)}$ from $N\left(\frac{n\bar{y}+\mu_{02}(1-(\rho^{(t)})^2)}{n+(1-(\rho^{(t)})^2)}, \frac{(\sigma_2^{(t)})^2(1-(\rho^{(t)})^2)}{n+(1-(\rho^{(t)})^2)}\right)$.

(g) Calculate $R(\theta^{(t)})$ using (5) and $Q(\theta^{(t)})$ using (3.13)

(h) set $t=t+1$.

(i) Repeat steps (b) to (h) 50,000 times

(j) Calculate the Bayes estimators for R using (3.19)-(3.20).

3. Repeat steps 1 and 2 for 500 times.
4. Calculate the bias and MSE of the estimators.

We repeat the simulation procedure for different values of μ_1, μ_2 and ρ by fixing $\sigma_1=1$ and $\sigma_2=1$. The bias and MSE of Bayes estimators for different combinations of μ_1, μ_2 and ρ for informative priors are given in table 2 and those of non-informative prior are given in table 3. We have obtained the HPD interval for R under SEL using non-informative prior. The AIL and CP for HPD interval are included in table 1. From the tables we have the following conclusions. The bias and MSE of all estimators decrease when the number of records n increases. Among the Bayes estimators, estimators under SEL function perform better than LL function in terms of bias and MSE. The bias and MSE of non-informative priors are smaller than that of informative priors. Also Bayes estimators under non-informative prior perform better than MLEs. The AILs of HPD intervals are smaller and the associated CPs are higher than that of percentile and BCa bootstrap confidence intervals.

5. Illustration Using Real Data

In this section, we illustrate the inferential procedures on $R=P(X<Y)$ developed in the previous sections using a real data. For that, we consider a study given in Hoffman and Johnson (2015) on water quality level of fresh water streams across the commonwealth of Virginia in USA. They compared the concentration levels of certain dissolved trace metals in freshwater to the worldwide standards using a well defined index function. The Virginia

Department of Environmental Quality (VDEQ) provided the data, which is available in the supplementary materials of Hoffman and Johnson (2015). The data set consist of concentration level of copper (Cu), lead (Pb), zinc (Zn) calcium (Ca) and magnesium (Mg) in freshwater streams of 184 independent probabilistic sites across Virginia. For the present study we take only one trace metal zinc and other two metals Ca and Mg which are used to find the worldwide quality standard of Zn. Since some observations on concentration levels of zinc are censored, we take the remaining 103 uncensored observations. Then the problem is to compare the metal concentration level of Zn in freshwater by its worldwide quality standard. The quantification of concentration level of Zn in freshwater is difficult and costly where as the quantification of other metals such as Ca and Mg are comparatively easy. Since water quality standard for Zn is a function of Ca and Mg, it can be determined easily. Therefore, we take water quality standard of Zn as X variate and concentration level of Zn in freshwater as Y variate. We want to estimate the probability that concentration level of Zn in freshwater is less than that of the worldwide quality standard of Zn in water. That is $P(Y < X)$, which is equivalent to $1 - P(X < Y)$.

Let X_1 and X_2 be the concentration level (mg/L) of Ca and Mg in fresh water then the water quality standard for Zn is given by

$$X = h^{0.8473} e^{0.884},$$

where h is the hardness factor ($mg/LCaCO_3$) from Ca and Mg and is given by

$$h = 2.497X_1 + 4.118X_2$$

Hoffman and Johnson (2015) assumes a 5 variate lognormal distribution to the original data. Here we fit lognormal distributions to both X and Y. The K-S test statistics for X and Y are 0.1713 and 0.1472 with corresponding p values are 0.2561 and 0.4307. Since X and Y follow lognormal distributions we take logarithm of X and Y to get the observations from normal distributions. The first 6 upper record values on X and its concomitants obtained from the data set are

n	**1**	**2**	**3**	**4**	**5**	**6**
R_n	0.127105	0.477121	0.737987	0.863323	1.079181	1.278754
$R_{[n]}$	1.53033	1.625011	1.594354	1.458808	1.836765	1.412407

We have obtained the MLE, Bayes estimators and CIs of $P(Y < X)$ based on first 6 record values on X variates and its concomitants on Y variates and are given below.

MLE	Bayes estimates		Confidence Interval		
	SEL	LL	percentile	BCa	HPD
0.9982	0.9853	0.9862	(0.9826,1)	(0.9812,0.9996)	(0.9781,0.9984)

For Bayes estimation we use non-informative priors for the parameters. Since the estimated values of $P(Y < X)$ is more or less 0.99 for all the cases, we can claim that the concentration level of zinc in freshwater across Virginia is relatively much lower than the worldwide quality standard of zinc in water.

6. Conclusion

In this work, we have considered the problem of estimation of $R = P(X < Y)$ for bivariate normal distribution using record values and its concomitants. The maximum likelihood and Bayesian estimators have been obtained for R. For obtaining the Bayes estimates, importance sampling method has been applied. Based on the simulation study we have concluded that among the estimators, Bayes estimators under squared error loss function perform better in terms of bias and MSE. Also Bayes estimators under non-informative priors perform better than the corresponding Bayes estimators under informative priors. AILs of HPD intervals are smaller and the associated CPs are higher than that of bootstrap confidence intervals.

Table 1: The bias & MSE of MLEs for R and AIL & CP for CIs.

ρ	n	μ_1	μ_2	R	MLE		Percentile CI		BCa CI		HPD	
					Bias	MSE	AIL	CP	AIL	CP	AIL	CP
-0.75	6	2	3	0.70351	0.14638	0.09531	0.32948	0.81	0.22546	0.87	0.18527	0.91
		3	2	0.29649	0.10152	0.07104	0.31968	0.83	0.23584	0.85	0.20718	0.92
		2	2	0.50000	0.09561	0.01731	0.32096	0.82	0.19574	0.86	0.11583	0.93
	8	2	3	0.70351	0.13979	0.07330	0.31639	0.83	0.27474	0.89	0.17791	0.92
		3	2	0.29649	0.09537	0.04839	0.31276	0.84	0.27212	0.87	0.15803	0.93
		2	2	0.50000	0.09360	0.01441	0.31291	0.82	0.27370	0.88	0.15640	0.92
	10	2	3	0.70351	0.12750	0.08491	0.32357	0.81	0.20150	0.89	0.13831	0.93
		3	2	0.29649	0.09255	0.03876	0.31152	0.82	0.17182	0.86	0.12990	0.94
		2	2	0.50000	0.08600	0.01267	0.32713	0.83	0.17000	0.88	0.13211	0.93
-0.5	6	2	3	0.71815	0.12965	0.08756	0.31822	0.82	0.20296	0.89	0.16759	0.94
		3	2	0.28185	0.14025	0.09298	0.32506	0.83	0.19256	0.86	0.15896	0.93

		2	2	0.50000	0.09776	0.10602	0.32105	0.84	0.24488	0.87	0.21645	0.94
	8	2	3	0.71815	0.12759	0.07954	0.31008	0.81	0.22149	0.89	0.16387	0.93
		3	2	0.28185	0.12615	0.08134	0.31704	0.82	0.21786	0.88	0.15692	0.94
		2	2	0.50000	0.10704	0.09096	0.31886	0.83	0.21801	0.87	0.15865	0.92
	10	2	3	0.71815	0.11998	0.05873	0.31710	0.84	0.21412	0.88	0.15625	0.94
		3	2	0.28185	0.10321	0.08599	0.32013	0.82	0.21196	0.87	0.13886	0.92
		2	2	0.50000	0.09409	0.03992	0.31535	0.81	0.21801	0.89	0.23921	0.92
-0.25	6	2	3	0.73646	0.12307	0.07141	0.32505	0.82	0.22226	0.86	0.25204	0.93
		3	2	0.26354	0.11849	0.04726	0.31084	0.82	0.21516	0.86	0.20207	0.94
		2	2	0.50000	0.10305	0.09433	0.33308	0.83	0.25140	0.88	0.20360	0.93
	8	2	3	0.73646	0.11645	0.05537	0.31538	0.84	0.27750	0.89	0.18913	0.94
		3	2	0.26354	0.10635	0.04612	0.32028	0.83	0.26761	0.89	0.16674	0.92
		2	2	0.50000	0.08736	0.07624	0.31187	0.84	0.27768	0.88	0.17065	0.93
	10	2	3	0.73646	0.09825	0.04940	0.31546	0.83	0.26493	0.87	0.15071	0.93
		3	2	0.26354	0.08861	0.03677	0.31096	0.84	0.26204	0.88	0.14041	0.94
		2	2	0.50000	0.08107	0.06178	0.32212	0.86	0.27768	0.91	0.14284	0.95
0.25	6	2	3	0.79289	0.10962	0.04223	0.32969	0.81	0.25004	0.88	0.15592	0.94
		3	2	0.20711	0.09786	0.05202	0.31250	0.82	0.25093	0.89	0.14870	0.93
		2	2	0.50000	0.09851	0.06973	0.32531	0.81	0.26319	0.87	0.14474	0.93
	8	2	3	0.79289	0.09829	0.07987	0.30616	0.80	0.27428	0.86	0.16836	0.94
		3	2	0.20711	0.07899	0.01804	0.30975	0.83	0.28907	0.88	0.15348	0.95
		2	2	0.50000	0.08506	0.04197	0.31119	0.82	0.25629	0.87	0.15840	0.94
	10	2	3	0.79289	0.09366	0.06232	0.32667	0.84	0.27698	0.88	0.14172	0.95
		3	2	0.20711	0.05964	0.02225	0.32250	0.82	0.24755	0.89	0.13861	0.94
		2	2	0.50000	0.07809	0.05033	0.31082	0.81	0.23629	0.86	0.12119	0.91
0.5	6	2	3	0.84134	0.09165	0.05251	0.31001	0.82	0.26343	0.87	0.13156	0.92
		3	2	0.15866	0.10792	0.06259	0.31945	0.84	0.25361	0.88	0.13166	0.91
		2	2	0.50000	0.08771	0.05989	0.32369	0.84	0.27377	0.89	0.14454	0.94
	8	2	3	0.84134	0.08427	0.06958	0.31121	0.81	0.26856	0.86	0.11919	0.93

		3	2	0.15866	0.10512	0.04869	0.30277	0.82	0.26953	0.86	0.14248	0.93
		2	2	0.50000	0.06125	0.05334	0.31291	0.82	0.29546	0.89	0.13891	0.93
	10	2	3	0.84134	0.08679	0.04042	0.30943	0.83	0.25444	0.88	0.13807	0.93
		3	2	0.15866	0.09490	0.03745	0.31025	0.82	0.28743	0.86	0.14285	0.92
		2	2	0.50000	0.05575	0.05196	0.30873	0.82	0.29546	0.87	0.12863	0.92
0.75	6	2	3	0.92135	-0.10132	0.09761	0.31630	0.81	0.22196	0.88	0.16117	0.94
		3	2	0.07865	0.10546	0.05130	0.30815	0.83	0.21022	0.89	0.12679	0.91
		2	2	0.50000	0.04208	0.03013	0.28871	0.82	0.21061	0.88	0.17279	0.93
	8	2	3	0.92135	0.10454	0.06106	0.31275	0.84	0.24982	0.89	0.14247	0.92
		3	2	0.07865	0.10471	0.04898	0.31608	0.83	0.20571	0.86	0.15861	0.94
		2	2	0.50000	-0.05195	0.02325	0.27457	0.81	0.24893	0.87	0.13900	0.91
	10	2	3	0.92135	0.10518	0.05960	0.36063	0.83	0.23591	0.88	0.14084	0.93
		3	2	0.07865	0.08260	0.03949	0.31352	0.84	0.22865	0.86	0.13138	0.93
		2	2	0.50000	-0.04950	0.05769	0.26600	0.82	0.23534	0.89	0.13241	0.91

Table 2: The bias and MSE for Bayes estimator for R under informative prior.

ρ	n	μ1	μ2	Prior I				Prior II			
				SEL		LL		SEL		LL	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-0.75	6	2	3	0.12963	0.08925	0.19625	0.09230	0.12853	0.08260	0.18528	0.08258
		3	2	-0.10245	0.01316	-0.10554	0.01517	-0.11610	0.01271	-0.11616	0.01572
		2	2	0.09304	0.01080	0.09299	0.01786	0.09268	0.01026	0.09260	0.01024
	8	2	3	0.11877	0.08456	0.18759	0.08951	0.10903	0.06556	0.17902	0.08549
		3	2	-0.09169	0.01293	-0.10631	0.01394	-0.08980	0.01024	-0.11995	0.01626
		2	2	0.07448	0.00983	0.08173	0.01980	0.07249	0.00908	0.07241	0.00686
	10	2	3	0.10280	0.08099	0.17962	0.08083	0.10105	0.07477	0.17031	0.07464
		3	2	-0.08953	0.01047	-0.10585	0.01478	-0.08115	0.00967	-0.11594	0.01676
		2	2	0.06838	0.00875	0.07978	0.01056	0.06340	0.00808	0.07598	0.00800

-0.5	6	2	3	0.12627	0.07011	0.16265	0.08101	0.12593	0.06805	0.15931	0.06803
		3	2	-0.11684	0.01206	-0.10172	0.01507	-0.11322	0.01174	-0.11328	0.01418
		2	2	0.08177	0.00920	0.09173	0.01083	0.07275	0.00837	0.07272	0.00636
	8	2	3	0.12509	0.06398	0.15088	0.07396	0.12474	0.06219	0.14737	0.06217
		3	2	-0.10222	0.01127	-0.10232	0.01267	-0.10122	0.01041	-0.12238	0.01642
		2	2	0.07925	0.00729	0.08916	0.01283	0.06442	0.00526	0.06436	0.00525
	10	2	3	0.10422	0.06071	0.14209	0.06608	0.10244	0.05133	0.13444	0.06124
		3	2	-0.09952	0.01023	-0.10000	0.01236	-0.08914	0.01007	-0.11409	0.01514
		2	2	0.07184	0.00721	0.06160	0.01272	0.06373	0.00537	0.06347	0.00532
-0.25	6	2	3	0.11240	0.05852	0.14027	0.07098	0.10359	0.05622	0.13589	0.05721
		3	2	-0.10276	0.01620	-0.10281	0.01960	-0.10990	0.01357	-0.10994	0.01428
		2	2	0.07221	0.00756	0.06219	0.01156	0.05850	0.00809	0.05848	0.00742
	8	2	3	0.10234	0.05521	0.13382	0.08520	0.09365	0.05104	0.12249	0.05111
		3	2	-0.09483	0.01372	-0.11158	0.01573	-0.08153	0.01290	-0.11823	0.01492
		2	2	0.06589	0.00730	0.06581	0.01087	0.05384	0.00653	0.05380	0.00685
	10	2	3	0.09232	0.05498	0.13206	0.05490	0.08776	0.04831	0.17690	0.04927
		3	2	-0.08113	0.01249	-0.11359	0.01494	-0.07597	0.01185	-0.11627	0.01484
		2	2	0.05705	0.00660	0.05684	0.00956	0.05171	0.00516	0.05151	0.00581
0.25	6	2	3	0.10585	0.04458	0.12106	0.06814	0.09811	0.04249	0.12111	0.04490
		3	2	-0.09327	0.01747	-0.12937	0.01948	-0.09130	0.01657	-0.13021	0.01758
		2	2	0.04283	0.00229	0.04281	0.00900	0.03646	0.00172	0.03645	0.00182
	8	2	3	0.09043	0.04405	0.12090	0.05421	0.08209	0.04106	0.12092	0.04405
		3	2	-0.08796	0.01745	-0.12807	0.01748	-0.08065	0.01675	-0.13815	0.01977
		2	2	0.03933	0.00202	0.05929	0.00802	0.03557	0.00160	0.03153	0.00190
	10	2	3	0.08043	0.04233	0.12043	0.05298	0.07864	0.03812	0.12021	0.04117
		3	2	-0.07189	0.01555	-0.11924	0.01562	-0.07830	0.01296	-0.13000	0.01835
		2	2	0.03402	0.00263	0.04005	0.00610	0.03386	0.00153	0.03376	0.00182

0.5	6	2	3	0.08728	0.04302	0.12070	0.06302	0.08666	0.04279	0.12067	0.04793
		3	2	-0.10503	0.02325	-0.15038	0.05326	-0.09984	0.02107	-0.15006	0.02309
		2	2	0.04313	0.00151	0.03311	0.00850	0.02619	0.00129	0.02618	0.00188
	8	2	3	0.08031	0.04149	0.12031	0.05149	0.08057	0.04247	0.12057	0.04546
		3	2	-0.09443	0.02149	-0.14438	0.04152	-0.08979	0.02043	-0.15395	0.02433
		2	2	0.03857	0.00144	0.03081	0.00643	0.02544	0.00119	0.02541	0.00915
	10	2	3	0.07638	0.03916	0.11963	0.04915	0.19655	0.03608	0.11965	0.03906
		3	2	-0.08751	0.01992	-0.13784	0.03936	-0.08445	0.01555	-0.14483	0.02163
		2	2	0.03208	0.00137	0.03194	0.00534	0.02499	0.00127	0.02491	0.00160
0.75	6	2	3	0.09142	0.04475	0.12114	0.08500	0.08929	0.04381	0.12129	0.04538
		3	2	-0.10489	0.03205	-0.17493	0.09107	-0.09396	0.03155	-0.17944	0.03257
		2	2	0.01679	0.00640	0.01678	0.00840	0.01570	0.00345	0.01569	0.00544
	8	2	3	0.08173	0.04336	0.12078	0.07335	0.08291	0.04151	0.12083	0.04350
		3	2	-0.08716	0.03198	-0.17724	0.08027	-0.08780	0.03068	-0.17804	0.03209
		2	2	0.01515	0.00183	0.01613	0.00942	0.01421	0.00131	0.01320	0.00314
	10	2	3	0.07078	0.03914	0.11970	0.05912	0.06987	0.03225	0.11986	0.03968
		3	2	-0.06593	0.02612	-0.15958	0.06220	-0.17690	0.02320	-0.17714	0.03208
		2	2	0.01314	0.00415	0.01709	0.00721	0.01290	0.00820	0.01395	0.00042

Table 3: The bias and MSE for Bayes estimator for *R* under non-informative prior.

ρ	**n**	**μ_1**	**μ_2**	**Non-informative**				**ρ**	**Non-informative**			
				SEL		**LL**			**SEL**		**LL**	
				Bias	**MSE**	**Bias**	**MSE**		**Bias**	**MSE**	**Bias**	**MSE**
-0.75	6	2	3	0.11282	0.08068	0.18230	0.08066	0.25	0.09183	0.04198	0.12098	0.04424
		3	2	-0.10020	0.01165	-0.12023	0.01847		-0.08139	0.01742	-0.13816	0.01961
		2	2	0.08136	0.00778	0.08132	0.00777		0.03218	0.00163	0.03920	0.00183
	8	2	3	0.10276	0.05690	0.17546	0.07687		0.08050	0.03242	0.12050	0.04242

		3	2	-0.07913	0.01637	-0.12142	0.01738		-0.07357	0.01517	-0.13573	0.01899
		2	2	0.07430	0.00683	0.07423	0.00818		0.03049	0.00160	0.03502	0.00161
	10	2	3	0.08134	0.07253	0.17162	0.07875		0.07200	0.03046	0.11998	0.04044
		3	2	-0.07129	0.00913	-0.12986	0.01921		-0.07137	0.01195	-0.13737	0.02000
		2	2	0.06475	0.00602	0.06444	0.00795		0.02917	0.00130	0.02908	0.00129
-0.5	6	2	3	0.12157	0.06694	0.15726	0.06785	0.5	0.07433	0.04135	0.12084	0.04354
		3	2	-0.10698	0.01087	-0.10705	0.01288		-0.09152	0.02069	-0.15218	0.02370
		2	2	0.07343	0.00746	0.07339	0.00846		0.02587	0.00118	0.02586	0.00088
	8	2	3	0.11825	0.06208	0.14754	0.06892		0.07203	0.04154	0.12034	0.04153
		3	2	-0.09512	0.01603	-0.12219	0.01911		-0.15651	0.01913	-0.15658	0.02515
		2	2	0.06635	0.00535	0.06628	0.00812		0.02187	0.00873	0.02584	0.00970
	10	2	3	0.10124	0.04643	0.13548	0.05637		0.19713	0.03921	0.11971	0.03920
		3	2	-0.08170	0.00961	-0.12047	0.01617		-0.14830	0.01845	-0.14854	0.02281
		2	2	0.06191	0.00510	0.06163	0.00704		0.02168	0.00109	0.02674	0.00113
-0.25	6	2	3	0.10156	0.04661	0.13557	0.05608	0.75	0.08114	0.03476	0.12114	0.04476
		3	2	-0.09205	0.01104	-0.11394	0.01405		-0.08604	0.03092	-0.18606	0.03493
		2	2	0.05405	0.00501	0.06402	0.00608		0.01473	0.00305	0.01473	0.00404
	8	2	3	0.09184	0.05025	0.12782	0.05251		0.06086	0.03136	0.12086	0.04363
		3	2	-0.07952	0.01032	-0.11803	0.01534		-0.08251	0.02825	-0.18257	0.03384
		2	2	0.05062	0.00367	0.05057	0.00465		0.01268	0.00813	0.01464	0.00909
	10	2	3	0.08515	0.04668	0.11451	0.05067		0.05803	0.04138	0.12030	0.04136
		3	2	-0.07194	0.01018	-0.11967	0.01622		-0.17755	0.02205	-0.17772	0.03210
		2	2	0.04498	0.00269	0.04483	0.00367		0.01176	0.00687	0.01424	0.00886

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