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Estimation of the Scale Parameter of Power Muth Distribution by U-Statistics

M.R. Irshad* R. Maya[1], N.K. Sajeevkumar[2] and S.P. Arun[3]
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ABSTRACT

In this work, we have considered a lifetime distribution namely power Muth distribution and derived the Best Linear Unbiased Estimator (BLUE) of the scale parameter of the power Muth distribution based on order statistics. We have further estimated the scale parameter of power Muth distribution by U-statistics based on best linear functions of order statistics as kernels.

1. Introduction

Muth (1977) revisited a continuous probability distribution with application in reliability theory. However this distribution has been overlooked in the literature until a recent article by Jodrá *et al.* (2015). A continuous random variable Z is said to have a Muth distribution with the shape parameter μ, if its probability density function (pdf) is given by

$$f(z;\mu) = (e^{\mu z}-\mu)\, e^{\{\mu z-\frac{1}{\mu}(e^{\mu z}-1)\}}, \; z>0,\ \mu\in(0,1]. \qquad (1.1)$$

The cumulative distribution function of Muth distribution is obtained as

$$F(z;\mu)=1-e^{\{\mu z-\frac{1}{\mu}(e^{\mu z}-1)\}}, \; z>0. \qquad (1.2)$$

From the view point of applications, the authors finally considered the scaled Muth distribution and fit this distribution to rainfall (in mm) data collected from the rain gauge station of Carrol, located in the State of New South Wales on the east coast of Australia and proved that the scaled Muth distribution performs well compared to the well-known life distributions like exponential, gamma, Lognormal and Weibull. A continuous random variable Z is said to have a Muth distribution with shape parameter μ and scale parameter β, if its pdf is given by

$$f(z;\mu,\beta)= \frac{1}{\beta}\left(e^{\frac{\mu}{\beta}z}-\mu\right) e^{\{\frac{\mu}{\beta}z-\frac{1}{\mu}\left(e^{\frac{\mu}{\beta}z}-1\right)\}}, \; z>0,\ \beta>0,\ \mu\in(0,1]. \qquad (1.3)$$

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*Corresponding author**: M.R. Irshad, Department of Statistics, Cochin University of Science and Technology,Kochi- 22, Kerala, E-mail: irshadm24@gmail.com.
[1]Author : R. Maya, Govt. College for Women, University of Kerala, Trivandrum-695 014. N.K.
[2]Auhtor : Sajeevkumar, Govt. College, Kariavattom, Trivandrum-685581.
[3]Author : S.P.Arun, Kerala University Library, Research Centre, University of Kerala, Trivandrum- 695 034.

The corresponding cumulative distribution function is given by

$$F(z;\mu,\beta) = 1\text{-}e^{\{\frac{\mu}{\beta}z-\frac{1}{\mu}\left(e^{\frac{\mu}{\beta}z}-1\right)\}},\ z>0. \qquad (1.4)$$

Jodrá *et al.* (2017) further studied this model and they developed power Muth distribution using the transformation X=$\beta Z^{1/\delta}$,where β>0 , δ>0 and Z follows the scaled Muth distribution with the shape parameter μ=1.

A continuous random variable X is said to have a power Muth distribution with shape parameterδ and scale parameter β, if its pdf is given by

$$f(x;\delta,\beta) =\frac{\delta}{\beta^{\delta}}x^{\delta-1}(e^{(\frac{x}{\beta})^{\delta}}-1)e^{\{(\frac{x}{\beta})^{\delta}-\left(e^{(\frac{x}{\beta})^{\delta}}-1\right)\}},\ x > 0. \qquad (1.5)$$

The corresponding cumulative distribution function is given by

$$F(x;\delta,\beta)=1\text{-}e^{\{(\frac{x}{\beta})^{\delta}-\left(e^{(\frac{x}{\beta})^{\delta}}-1\right)\}},\ x > 0. \qquad (1.6)$$

The introduction of the new parameter δ leads to a larger class of probability distributions for non-negative random variables with a wide range of values for the asymmetry and kurtosis coefficients, increasing generalized failure rate as well as increasing or bathtub shape failure rate. Finally the authors showed that the supremacy of the model through illustrating two real life data sets. Even though the inferential aspects of this distribution are concerned there is a necessity to derive some estimators for the scale parameter of power Muth distribution.

When we consider the classical estimation techniques like method of moments and method of maximum likelihood obviously we face some difficulties. In order to obtain moment type estimators we need moments of the corresponding distributions in closed form. But in the case of power Muth distribution the moments are obtained only in terms of the generalized integro-exponential function (see, Milgram, 1985). Also in general, maximum likelihood estimation (MLE) ends up with some limitations. When the sample size is small, MLE is not even unbiased in many cases. Though the expression for the asymptotic variance of the MLE's can be obtained, the exact variance of the small sample cases is generally not explicitly available. In the case of power Muth distribution, the maximum likelihood estimators of the parameters are not in the closed form and it can be obtained only by numerical optimization technique like Newton-Rashpon method and Fisher's scoring algorithm using some statistical software's.

Order statistics enter into the statistical inference problems in different ways. Best linear unbiased estimators of the location and scale parameters of location scale family of distributions (see, Llyod, 1952) by order statistics is well-known and extensively discussed in the available literature. Moreover the estimates obtained in this case are optimal in the scene that their variances are least among all other linear functions of order statistics which are used to estimate unbiasedly the required parameters. Motivated by the applications of ordered random variables, in this work our aim is derive some estimators of scale parameter of power Muth distribution using ordered random variables.

2. Best Linear Unbiased Estimator of Scale Parameter of power Muth Distribution Using Order Statistics

The probability density function of the power Muth distribution is given in (1.5) belongs to the scale family of distributions having the probability density functions of the form $f(x;\beta) = \frac{1}{\beta} f_0\left(\frac{x}{\beta}\right), \beta > 0.$ Lloyd (1952) has introduced a method of estimating the scale parameter involved in the scale family of distribution by the best linear function of order statistics arising from it . There is extensive literature available on the problem of estimation of by order statistics. For example see, David and Nagaraja (2003).

Till at this time, not any work is seen attempted in the available literature to estimate the scale parameter β involved in (1.5). Hence in this section, our aim is to estimate the scale parameter β included in (1.5) for some known values of the shape parameter δ using order statistics.

Let X $=(X_{1:m,}X_{2:m,} \dots, X_{m:m})'$ be the vector of order statistics of a random sample of size m drawn from (1.5). Define $Y_{r:m}=\frac{X_{r:m}}{\beta}$, r= 1,2,….,m. Then $Y_{r:m}$, r= 1,2,…,m are distributed as the order statistics of a random sample of size m drawn from standard form of (1.5) with pdf $f_0(y)$.
Let $\boldsymbol{\alpha} =(\alpha_{1:m,}\alpha_{2:m,} \dots, \alpha_{m:m})'$ and V = $((V_{r,s:m}))$ be the vector of means and dispersion matrix of the vector of order statistics of a random sample of size m drawn from $f_0(y)$. Then the best linear unbiased estimator (BLUE) of β involved in (1.5) namely $\breve{\beta}$ based on order statistics is given by

$$\breve{\beta}= \frac{\alpha' V^{-1}}{(\alpha' V^{-1}\alpha)} X, \tag{2.1}$$

with variance given by

$$\text{var}(\breve{\beta})=\frac{\boldsymbol{\beta^2}}{(\alpha' V^{-1}\alpha)}. \tag{2.2}$$

From (2.1), we have $\breve{\beta}$ is a linear functions of order statistics $X_{r:m}$, r=1,2,…,m and hence $\breve{\beta}$ can be written as

$\breve{\beta}= \sum_{r=1}^{m} e_{r:m} X_{r:m}$, where $e_{r:m}$, r=1,2,…,m are constants. (2.3)

We have computed the values of means, variances and covariance's of entire order statistics of a sample of size m arising from the standard form of (1.5) for m=2(1) 10 and δ=0.25(0.25) 0.75 by using MATHCAD software and are presented in Table 1and table 2. Using these values we have evaluated the coefficients of $X_{r:m}$ in the BLUE of β and its variances for m=2(1) 10 and δ=0.25(0.25)0.75 by using MATHCAD software and presented in Table 3.

Table 1: Means of order statistics arising from standard from of power Muth distribution for m=2(1)10 and δ=0.25(0.25)0.75.

M	**r**	**δ=0.25**	**δ =0.50**	**δ =0.75**
2	1	0.78956	0.68066	0.71397
	2	3.85319	1.70473	1.37428
3	1	0.40698	0.48327	0.56643
	2	1.55470	1.07545	1.00905
	3	5.00244	2.01936	1.55690
4	1	0.25110	0.37686	0.47891
	2	0.87464	0.80251	0.82899
	3	2.23477	1.34839	1.18911
	4	5.92499	2.24302	1.67950
5	1	0.17149	0.30982	0.41967
	2	0.56954	0.64503	0.71589
	3	1.33231	1.03872	0.99863
	4	2.83640	1.55484	1.31610
	5	6.69714	2.41507	1.77035
6	1	0.12507	0.26352	0.37633
	2	0.40359	0.54130	0.63639

	3	0.90142	0.85249	0.87487
	4	1.76319	1.22495	1.12239
	5	3.37301	1.71978	1.41295
	6	7.36197	2.55413	1.84183
7	1	0.09551	0.22955	0.34294
	2	0.03240	0.46736	0.57666
	3	0.65657	0.72613	0.78574
	4	1.22790	1.02097	0.99372
	5	2.16467	1.37794	1.21889
	6	3.85635	1.85651	1.49057
	7	7.94623	2.67040	1.90038
8	1	0.07547	0.20351	0.31626
	2	0.23577	0.41181	0.52971
	3	0.50229	0.63404	0.71750
	4	0.91370	0.87961	0.89946
	5	1.54209	1.16233	1.08799
	6	2.53821	1.50731	1.29743
	7	4.29573	1.97291	1.55495
	8	8.46774	2.77004	1.94972
9	1	0.06123	0.18289	0.29434
	2	0.18940	0.36843	0.49160
	3	0.39808	0.56362	0.66310
	4	0.71070	0.77487	0.82632
	5	1.16745	1.01054	0.99088
	6	1.84180	1.28376	1.16568

	7	2.88641	1.61909	1.36331
	8	4.69839	2.07400	1.60970
	9	8.93890	2.85704	1.99223
10	1	0.05073	0.16615	0.27594
	2	0.15574	0.33357	0.45989
	3	0.32405	0.50787	0.61841
	4	0.57081	0.69371	0.76736
	5	0.92055	0.89662	0.91476
	6	1.41435	1.12445	1.06700
	7	2.12677	1.38996	1.23146
	8	3.21197	1.71728	1.41981
	9	5.06999	2.16319	1.65717
	10	9.36878	2.93414	2.02945

Table 2: Variances and co-variances $V_{r,s:n}$ of order statistics arising from standard form of power Muth distribution for $1 \leq r \leq s \leq m$, m = 2(1)10 and δ = 0.25(0.25)0.75.

m	**R**	**s**	**δ=0.25**	**δ=0.05**	**δ=0.75**
2	1	1	1.72800	0.32624	0.17588
	1	2	2.35063	0.26218	0.10900
	2	2	19.11475	0.94982	0.29177
3	1	1	0.49820	0.17343	0.11584
	1	2	0.66672	0.14886	0.07891
	1	3	0.86532	0.12238	0.05361
	2	2	3.30942	0.39809	0.16536
	2	3	4.19924	0.32896	0.11366

	3	3	23.04576	0.92869	0.25494
4	1	1	0.19996	0.10907	0.08520
	1	2	0.26654	0.09629	0.06038
	1	3	0.33674	0.08430	0.04500
	1	4	0.42196	0.07037	0.03211
	2	2	1.10129	0.23062	0.11584
	2	3	1.37301	0.20238	0.08687
	2	4	1.70449	0.16930	0.06229
	3	3	4.59259	0.41656	0.15003
	3	4	5.60311	0.35000	0.10859
	4	4	25.78238	0.89931	0.22979
5	1	1	0.09690	0.07550	0.06675
	1	2	0.12908	0.06776	0.04838
	1	3	0.16150	0.06089	0.03746
	1	4	0.19658	0.05398	0.02922
	1	5	0.24008	0.04555	0.02152
	2	2	0.48544	0.15347	0.08879
	2	3	0.60180	0.13810	0.06902
	2	4	0.72805	0.12256	0.05400
	2	5	0.88503	0.10354	0.03986
	3	3	1.67597	0.25335	0.10844
	3	4	2.00777	0.22528	0.08524
	3	5	2.42216	0.19069	0.06318
	4	4	5.63207	0.41882	0.13744
	4	5	6.69846	0.35588	0.10267

	5	5	27.82845	0.87143	0.21160
6	1	1	0.05308	0.05562	0.05450
	1	2	0.07073	0.05048	0.04010
	1	3	0.08812	0.04605	0.03172
	1	4	0.10605	0.04184	0.02566
	1	5	0.12580	0.03741	0.02059
	1	6	0.15071	0.03183	0.01550
	2	2	0.25138	0.11059	0.07166
	2	3	0.31097	0.10097	0.05686
	2	4	0.37256	0.09180	0.04609
	2	5	0.44052	0.08214	0.03704
	2	6	0.52632	0.06993	0.02792
	3	3	0.78834	0.17468	0.08513
	3	4	0.93816	0.15900	0.06923
	3	5	1.10393	0.14241	0.05577
	3	6	1.31371	0.12137	0.04212
	4	4	2.19228	0.26266	0.10112
	4	5	2.55969	0.23566	0.08177
	4	6	3.02645	0.20120	0.06199
	5	5	6.48814	0.41528	0.12745
	5	6	7.57873	0.35579	0.09729
	6	6	29.43376	0.84664	0.19778
7	1	1	0.03170	0.04282	0.04580
	1	2	0.04227	0.03918	0.03406
	1	3	0.05255	0.03610	0.02733

	1	4	0.06289	0.03326	0.02257
	1	5	0.07373	0.03045	0.01876
	1	6	0.08583	0.02741	0.01535
	1	7	0.10134	0.02347	0.01174
	2	2	0.14470	0.08397	0.05984
	2	3	0.17888	0.07742	0.04812
	2	4	0.21330	0.07136	0.03980
	2	5	0.24943	0.06536	0.03311
	2	6	0.28983	0.05885	0.02711
	2	7	0.34146	0.05041	0.02076
	3	3	0.42847	0.12930	0.07000
	3	4	0.50838	0.11928	0.05802
	3	5	0.59238	0.10931	0.04835
	3	6	0.68647	0.09848	0.03965
	3	7	0.80686	0.08442	0.03040
	4	4	1.08164	0.18551	0.08059
	4	5	1.25383	0.17071	0.06732
	4	6	1.44712	0.15345	0.05531
	4	7	1.69506	0.13167	0.04250
	5	5	2.64917	0.26591	0.09479
	5	6	3.03799	0.24014	0.07813
	5	7	3.53838	0.20639	0.06023
	6	6	7.20607	0.40958	0.11943
	6	7	8.30426	0.35316	0.09263
	7	7	30.73770	0.82485	0.18684

8	1	1	0.02018	0.03406	0.03934
	1	2	0.02694	0.03136	0.02950
	1	3	0.03346	0.02911	0.02391
	1	4	0.03991	0.02707	0.02001
	1	5	0.04651	0.02511	0.01696
	1	6	0.05350	0.02312	0.01435
	1	7	0.06141	0.02091	0.01191
	1	8	0.07167	0.01800	0.00923
	2	2	0.08981	0.06619	0.05119
	2	3	0.11102	0.06146	0.04156
	2	4	0.13205	0.05717	0.03482
	2	5	0.15355	0.05305	0.02954
	2	6	0.17638	0.04886	0.02502
	2	7	0.20218	0.04421	0.02077
	2	8	0.23547	0.03806	0.01611
	3	3	0.25609	0.10028	0.05933
	3	4	0.30341	0.09333	0.04979
	3	5	0.35182	0.08665	0.04229
	3	6	0.40329	0.07983	0.03585
	3	7	0.46151	0.07226	0.02979
	3	8	0.53672	0.06225	0.02312
	4	4	0.60999	0.13997	0.06710
	4	5	0.70460	0.13003	0.05709
	4	6	0.80533	0.11987	0.04846
	4	7	0.91945	0.10856	0.04032

	4	8	1.06708	0.09357	0.03133
	5	5	1.35586	0.19107	0.07631
	5	6	1.54315	0.17628	0.06491
	5	7	1.75576	0.15978	0.05410
	5	8	2.03138	0.13785	0.04211
	6	6	3.05305	0.26619	0.08942
	6	7	3.45470	0.24160	0.07475
	6	8	3.97688	0.20876	0.05836
	7	7	7.81819	0.40319	0.11286
	7	8	8.91483	0.34943	0.08862
	8	8	31.82491	0.80567	0.17793
9	1	1	0.01351	0.02778	0.03436
	1	2	0.01805	0.02572	0.02593
	1	3	0.02240	0.02400	0.02118
	1	4	0.02668	0.02247	0.01789
	1	5	0.03097	0.02102	0.01536
	1	6	0.03540	0.01960	0.01325
	1	7	0.04016	0.01813	0.11360
	1	8	0.04557	0.01647	0.00953
	1	9	0.05266	0.01424	0.00747
	2	2	0.05899	0.05366	0.04460
	2	3	0.07294	0.05010	0.03648
	2	4	0.08664	0.04691	0.03085
	2	5	0.10042	0.04391	0.02650
	2	6	0.11465	0.04095	0.02287

	2	7	0.12990	0.03787	0.01962
	2	8	0.14729	0.03441	0.01647
	2	9	0.16992	0.02975	0.01291
	3	3	0.16381	0.08041	0.05137
	3	4	0.19395	0.07532	0.04351
	3	5	0.22428	0.07053	0.03741
	3	6	0.25564	0.06579	0.03231
	3	7	0.28927	0.06087	0.02774
	3	8	0.32764	0.05531	0.02329
	3	9	0.37759	0.04784	0.01826
	4	4	0.37548	0.11027	0.05747
	4	5	0.43289	0.10330	0.04948
	4	6	0.49230	0.09640	0.04277
	4	7	0.55610	0.08921	0.03675
	4	8	0.62894	0.08110	0.03089
	4	9	0.72389	0.07018	0.02423
	5	5	0.78722	0.14625	0.06410
	5	6	0.89249	0.13657	0.05549
	5	7	1.00567	0.12645	0.04774
	5	8	1.03509	0.11500	0.04016
	5	9	1.30399	0.09958	0.03155
	6	6	1.60867	0.19375	0.07250
	6	7	1.80623	0.17953	0.06248
	6	8	2.03253	0.16340	0.05265
	6	9	2.32843	0.14161	0.04142

	7	7	3.41151	0.26493	0.08487
	7	8	3.82059	0.24143	0.07170
	7	9	4.35681	0.20953	0.05656
	8	8	8.34763	0.39670	0.10736
	8	9	9.43752	0.34526	0.08514
	9	9	32.75009	0.78866	0.17050
10	1	1	0.00940	0.02312	0.03041
	1	2	0.01258	0.02150	0.02308
	1	3	0.01561	0.02015	0.01896
	1	4	0.01857	0.18960	0.01613
	1	5	0.02151	0.01785	0.01397
	1	6	0.02450	0.01679	0.01220
	1	7	0.02762	0.15710	0.01066
	1	8	0.03097	0.01458	0.00924
	1	9	0.03484	0.01329	0.00782
	1	10	0.03997	0.01153	0.00618
	2	2	0.04050	0.04447	0.03942
	2	3	0.05010	0.04170	0.03243
	2	4	0.05946	0.03924	0.02762
	2	5	0.06878	0.03696	0.02394
	2	6	0.07824	0.03475	0.02091
	2	7	0.08810	0.03254	0.01827
	2	8	0.09875	0.03020	0.01584
	2	9	0.11098	0.02753	0.01341
	2	10	0.12702	0.02389	0.01060

	3	3	0.11030	0.06612	0.04522
	3	4	0.13055	0.06224	0.03855
	3	5	0.15072	0.05864	0.03344
	3	6	0.17122	0.05515	0.02923
	3	7	0.19259	0.05165	0.02556
	3	8	0.21568	0.04794	0.02216
	3	9	0.24221	0.04371	0.01877
	3	10	0.27699	0.03794	0.01484
	4	4	0.24606	0.08958	0.05021
	4	5	0.28337	0.08442	0.04359
	4	6	0.32132	0.07942	0.03813
	4	7	0.36091	0.07439	0.03336
	4	8	0.40373	0.06907	0.02894
	4	9	0.45295	0.06299	0.02453
	4	10	0.51753	0.05470	0.01940
	5	5	0.49623	0.11661	0.05532
	5	6	0.56134	0.10975	0.04844
	5	7	0.62932	0.10284	0.04242
	5	8	0.70290	0.09551	0.03682
	5	9	0.78757	0.08713	0.03123
	5	10	0.89874	0.07569	0.02472
	6	6	0.95628	0.14995	0.06129
	6	7	1.06933	0.14057	0.05373
	6	8	1.09184	0.13062	0.04670
	6	9	1.33297	0.11921	0.03964

	6	10	1.51852	0.10361	0.03141
	7	7	1.84057	0.19475	0.06915
	7	8	2.04514	0.18109	0.06018
	7	9	2.28121	0.16539	0.05116
	7	10	2.59208	0.14387	0.04060
	8	8	3.73147	0.26286	0.08096
	8	9	4.14451	0.24034	0.06898
	8	10	4.68957	0.20936	0.05488
	9	9	8.81122	0.39039	0.10269
	9	10	9.89141	0.34099	0.08210
	10	10	33.55036	0.77347	0.16417

3. Estimation of the Scale Parameter of Power Muth Distribution Using U–Statistics

The main disadvantage involved in the Lloyd's BLUE of scale parameter of a distribution by order statistics is that, in order to obtain the estimator one needed the values of means, variances, and co-variances of the entire order statistics arising from the standard form of the original distribution. This makes the method unsuitable to applied statisticians. We all know, most of the distributions which belongs to scale family, explicit expressions does not exist for the means, variances and co-variances of these order statistics. In this case these values are usually calculated numerically for samples sizes up to 20 and the estimators and their variances are obtained for samples sizes up to 20. However, if one obtain the BLUE of the scale parameter β by order statistics based on a moderate sample size m and use it as kernel of degree m to construct a suitable U-statistic to estimate β, then these U-statistics would be highly useful as they estimate the parameter explicitly. Moreover this estimator is highly preferred as they hold the optimal conditions of BLUE as well as U-statistics. The U-statistics obtained in this method are distributed asymptotically as normal and hence those U-statistics can be even used for testing of hypothesis problem on the scale parameter of power Muth distribution for large sample sizes. Hence in this section we estimate the scale parameter β of power Muth distribution using U-statistics based on the best linear functions of order statistics as kernels. Thomas and Sreekumar (2008) developed a method of estimation by U- statistics using best linear unbiased estimators of the location and scale parameters of a distribution based on order statistics with an appropriate small sample size as kernels. Some works related with U-statistics of this type are seen in the available literature. For more details see, Irshad and Sajeev kumar (2014), Thomas and Priya (2015) and Irshad and Maya (2018).

3.1 U-Statistics

Let X_1, X_2 , …, X_n be independent samples arising from a population with cumulative distribution function F(x;θ). The U-statistic for the estimable parameter θ corresponding to the symmetric kernel h(.) of degree m is given by

$$U(X_1, X_2, \ldots, X_n) = \frac{1}{\binom{n}{m}} \sum_{\alpha \in A} h(X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_m}) \quad (3.1)$$

Where A = {α|α = $(\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\alpha_1 < \alpha_2 < \ldots < \alpha_m$} is one of the $\binom{n}{m}$ combinations of m integers chosen without replacement from the set (1,2,…,n).

Suppose that

$$E[h(X_1, X_2, \ldots, X_m)] = \theta \text{ and } E[h^2(X_1, X_2, \ldots, X_m)] < \infty. \quad (3.2)$$

Let h($X_1, X_2, \ldots, X_\omega, X_{\omega+1}, \ldots, X_m$) and h($X_1, X_2, \ldots, X_\omega, X_{m+1}, \ldots, X_{2m-\omega}$) be two random variables having exactly ω sample observations in common , ω= 1,2,…,m . Let $\varsigma_\omega^{(m)}$ be the covariance between these two random variables. Then the variance of the U- statistic given in (3.1) is given by (see, Hoeffding (1948))

$$\mathrm{Var}[U(X_1, X_2, \ldots, X_n)] = \frac{1}{\binom{n}{m}} \sum_{\omega=1}^{m} \binom{m}{\omega} \binom{n-m}{m-\omega} \varsigma_\omega^{(m)} . \quad (3.3)$$

Clearly the U-statistics defined in (3.1) is an unbiased estimator of θ. For more details about the desirable properties of U- statistics (see, Serfling (1980)).

3.2 U- Statistics Based on BLUE of β as Kernel

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m arising from (1.5) . As a consequence of (2.3) , $\check{\beta}$ can be represented as

$$h(X_1, X_2, \ldots, X_m) = e_{1:m}X_{1:m} + e_{1:m}X_{2:m} + \ldots + e_{m:m}X_{m:m}, \quad (3.4)$$

where $e_{1:m}$, $e_{2:m}$, … , $e_{m:m}$ are suitable constants . Now by considering h (X_1, X_2 , … , X_m) as a kernel of degree m , using the results of Thomas and Sreekumar (2008) , the U- statistic based on kernel (3.4) is given by

$$U_n^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i} \binom{r-1}{i} e_{i+1:m} \right] X_{r:n} \, . \qquad (3.5)$$

In the above equation (3.5), for any two non-negative integers p and q , unlike the usual definition of $\binom{p}{q}$ for q≤ p we also define $\binom{p}{q}$ = 0 for p<q.

Let ς_d^m = Cov $\left[h(X_1, X_2, \ldots, X_d, X_{d+1}, \ldots, X_m),\ h(X_1, X_2, \ldots, X_d, X_{m+1}, \ldots, X_{2m-d}\right]$, for d= 1,2,3,….,m , then the variance of $U_n^{(m)}$ is given by ,

$$\operatorname{var}\left[U_n^{(m)}\right] = \frac{1}{\binom{n}{m}} \sum_{d=1}^{m} \binom{m}{d} \binom{n-m}{m-d} \varsigma_d^{(m)} . \qquad (3.6)$$

Clearly $\varsigma_m^{(m)}$ = Var$\left[h(X_1, X_2, \ldots, X_m)\right]$ and is given in (2.2). Now we evaluate the values of $\varsigma_d^{(m)}$ for d= 1,2,…, m-1 , using the methodology developed by Thomas and Sreekumar (2008) as explained in the following steps.

Define the vector b_{m+k} for k=1,2,…,m-1 as

$$b'_{m+k} = \left[\frac{\sum_{i=0}^{m-1} \binom{m+k-1}{m-1-i} \binom{0}{i} e_{i+1:m}}{\binom{m+k}{m}}, \frac{\sum_{i=0}^{m-1} \binom{m+k-2}{m-1-i} \binom{1}{i} e_{i+1:m}}{\binom{m+k}{m}}, \ldots, \frac{\sum_{i=0}^{m-1} \binom{0}{m-1-i} \binom{m+k-1}{i} e_{i+1:m}}{\binom{m+k}{m}} \right]. \qquad (3.7)$$

Then, the expression for $U_{m+k}^{(m)}$ for k=1,2,…,m-1 is obtained by (putting n = m+k in (3.5))

$$U_{m+k}^{(m)} = b'_{m+k} X_{m+k} \, , \qquad (3.8)$$

where $X_{m+k} = \left(X_{1:m+k}, X_{2:m+k}, \ldots, X_{m+k:m+k}\right)'$.

Hence ,

$$\operatorname{Var}\left[U_{m+k}^{(m)}\right] = (b'_{m+k} V_{m+k} b_{m+k}) \beta^2 \, . \qquad (3.9)$$

Where V_{m+k} is the dispersion matrix of the vector of order statistics of random sample of size m+k arising from the distribution with pdf f_0(y). Consequently, we can write the following matrix equation (see, Thomas and Sreekumar (2008)),

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \binom{m}{m-1}\binom{1}{1} \\ 0 & 0 & \cdots & \binom{m}{m-2}\binom{2}{2} & \binom{m}{m-1}\binom{2}{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{m}{1}\binom{m-1}{m-1} & \binom{m}{2}\binom{m-1}{m-2} & \cdots & \binom{m}{m-2}\binom{m-1}{2} & \binom{m}{m-1}\binom{m-1}{1} \end{bmatrix} \times$$

$$\begin{bmatrix} \varsigma_1^{(m)} \\ \varsigma_2^{(m)} \\ \cdot \\ \cdot \\ \cdot \\ \varsigma_{m-1}^{(m)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_{m-1} \end{bmatrix}. \qquad (3.10).$$

Where $z_k = \binom{m+k}{m}(b'_{m+k}V_{m+k})\beta^2 - \varsigma_m^{(m)}$, k= 1,2,…,m-1. If we write H to denote the coefficient matrix of the left side of (3.10) and W denote the vector in the right side of (3.10), we have

$$\begin{bmatrix} \varsigma_1^{(m)} \\ \varsigma_2^{(m)} \\ \cdot \\ \cdot \\ \cdot \\ \varsigma_{m-1}^{(m)} \end{bmatrix} = H^{-1}\text{W}. \qquad (3.11)$$

Once we obtain the values of $\varsigma_d^{(m)}$, d=1,2,..,m-1 , then the actual variances of the U-statistics for estimating β based on any sample of size n can be obtained by using (3.5) without any further direct evaluation of moments of order statistics.

The main peculiarity of this method is that if one uses the BLUE based on sample of size m as kernel, then the evaluation of variances and co-variances of order statistics of sample of size up to 2m-1 arising from the standard form of (1.5) alone are necessary to obtain the explicit expression for the variance of the U-statistic$U_n^{(m)}$ whatever is the sample size, however large it may be.

For example it for a given value of the shape parameter δ , we use $\check{\beta}$ as given in (2.1) for m=5, then with the computation of moments of order statistics arising from the standard form the power Muth distribution defined in (1.5) for sample size up to 9, one can obtain the exact form of appropriate U-statistics estimator of β and their variances for any sample of size n. Using the values of variance and co-variances of order statistics (given in Table 2) and the coefficients of BLUE of β (given in Table 3), we have obtained the values of $\varsigma_d^{(m)}$ for d=1,2,…,m-1, m=2 (1) 5 and δ= 0.25 (0.25) 0.75 and are given in Table 4. Using the values of $\varsigma_d^{(m)}$ we have evaluated the numerical values of the variances of the U-statistic estimator, Var ($U_n^{(m)}$) for m = 2(1)5, n = 5(5)20(10)40(20)100 and for δ = 0.25 (0.25)0.75 and are given in Table 5.

Table 4: Values of $\varsigma_d^{(m)}$, for d = 1,2,…,m and m=2(1)5.

m	**d**	**δ = 0.25**	**δ = 0.50**	**δ =0.75**
		$\boldsymbol{\beta^{-2}\varsigma_d^{(m)}}$	$\boldsymbol{\beta^{-2}\varsigma_d^{(m)}}$	$\boldsymbol{\beta^{-2}\varsigma_d^{(m)}}$
2	1	0.58987	0.15200	0.07097
	2	1.18203	0.30926	0.14770
3	1	0.26165	0.06672	0.03046
	2	0.52388	0.13459	0.06226
	3	0.78682	0.20371	0.09550
4	1	0.14696	0.03736	0.01685
	2	0.29427	0.07493	0.03419
	3	0.44176	0.11306	0.05199
	4	0.58951	0.15164	0.07032
5	1	0.09399	0.02278	0.01068
	2	0.18815	0.04763	0.02160
	3	0.28240	0.07182	0.03269
	4	0.37675	0.09615	0.04402
	5	0.47127	0.12069	0.05558

Table 5: Var ($U_n^{(m)}$) for δ = 0.25 (0.25) 0.75 and for m = 2(1)5.

m	**n**	**δ = 0.25**	**δ = 0.50**	**δ = 0.75**
		$\frac{Var(U_n^{(m)})}{\beta^2}$	$\frac{Var(U_n^{(m)})}{\beta^2}$	$\frac{Var(U_n^{(m)})}{\beta^2}$
2	5	0.47213	0.12213	0.05735
	10	0.23600	0.06092	0.02852
	15	0.15732	0.04058	0.01898
	20	0.11799	0.03043	0.01422
	30	0.07865	0.02028	0.00948
	40	0.05899	0.01521	0.00710
	60	0.03933	0.01014	0.00473
	80	0.02949	0.00760	0.00355
	100	0.02360	0.00608	0.00284
3	5	0.47150	0.12110	0.05600
	10	0.23560	0.06028	0.02768
	15	0.15704	0.04013	0.01839
	20	0.11777	0.03008	0.01377
	30	0.07851	0.02004	0.00917
	40	0.05888	0.01503	0.00687
	60	0.03930	0.01000	0.00460
	80	0.02944	0.00751	0.00343
	100	0.02355	0.00601	0.00274
4	5	0.47131	0.12078	0.05566
	10	0.25339	0.05999	0.02735
	15	0.15687	0.03993	0.01814
	20	0.11763	0.02993	0.01357

	30	0.07841	0.01994	0.00903
	40	0.05880	0.01495	0.00676
	60	0.03920	0.00997	0.00450
	80	0.02940	0.00747	0.00338
	100	0.02352	0.00598	0.00270
5	5	0.47127	0.12069	0.05558
	10	0.23530	0.05968	0.02719
	15	0.15680	0.03940	0.01802
	20	0.11757	0.02934	0.01347
	30	0.07836	0.01940	0.00895
	40	0.05876	0.01448	0.00671
	60	0.03917	0.00960	0.00446
	80	0.02938	0.00718	0.00335
	100	0.02350	0.00574	0.00267

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Table 3: Coefficients of $X_{r:m}$ in the BLUE $\breve{\beta}$ and $V_1 = \frac{\mathrm{Var}(\breve{\beta})}{\beta^2}$.

n	**δ**	**coefficients**										$\boldsymbol{V_1}$
		$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$	$\boldsymbol{e_8}$	$\boldsymbol{e_9}$	$\boldsymbol{e_{10}}$	
2		0.25935	0.20638									1.18203
3		0.19562	0.15977	0.13433								0.78682
4		0.16090	0.13339	0.11413	0.09922							0.58951
5		0.13843	0.11614	0.10025	0.08835	0.07853						0.47127
6	0.25	0.12241	0.10374	0.09016	0.07995	0.07191	0.06493					0.39251
7		1.04723	-1.13658	0.44043	0.05317	0.04754	0.04319	0.03937				0.23919
8		0.10024	0.08692	0.07624	0.06815	0.06182	0.05661	0.05225	0.04817			0.29416
9		0.05866	0.09143	0.07151	0.10018	0.00705	0.07065	0.04987	0.04564	0.04281		0.26204
10		0.08501	0.07578	0.06695	0.06012	0.05481	0.05047	0.04684	0.04369	0.04096	0.03826	0.23521
2		0.25594	0.48441									0.30926
3		0.14220	0.20728	0.35079								0.20371
4		0.09381	0.13130	0.17420	0.27837							0.15164
5		0.06776	0.09338	0.11878	0.15072	0.23231						0.12069

Table 3: Continued

6	0.5	0.01428	0.09050	0.08461	0.10947	0.13333	0.20035					0.10028
7		0.04106	0.05640	0.07021	0.08222	0.09847	0.11985	0.17641				0.08565
8		0.03350	0.04622	0.05659	0.06678	0.07776	0.09082	0.10877	0.15800			0.07477
9		0.02782	0.03876	0.04736	0.05538	0.06369	0.07301	0.08423	0.09986	0.14331		0.06634
10		0.02366	0.03303	0.04150	0.04492	0.05451	0.06107	0.06812	0.07885	0.09241	0.13130	0.05962
2		0.21917	0.61379									0.14770
3		0.10746	0.21069	0.46650								0.09550
4		0.06481	0.12019	0.19220	0.38153							0.07032
5		0.04364	0.07913	0.11952	0.17482	0.32513						0.05558
6	0.75	0.03146	0.05659	0.08309	0.11506	0.16000	0.28463					0.04591
7		0.02603	0.04728	-0.04905	-0.18312	0.70381	0.33030	-0.08728				0.04251
8		0.01847	0.03352	0.04798	0.0635	0.08157	0.17420	0.13678	0.2299			0.03403
9		0.01780	0.02696	0.03853	0.04877	0.06692	0.07766	0.09903	0.12762	0.21045		0.03012
10		0.01202	0.02224	0.03161	0.04130	0.05141	0.06291	0.07669	0.09429	0.11987	0.19419	0.02702