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The Holistic and Generalized (H-G) Family of Continuous Semi-Bounded Distribution

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ABSTRACT

This paper proposes a new Holistic and Generalized Continuous Semi-bounded distribution which is also called as H-G family of Semi-bounded distribution. The density function of the proposed distribution is expressed in terms of Kummer's U function. The cumulative distribution function and the constants such as Central moments, Non-central moments, Skewness, kurtosis, Shannon's differential entropy are also computed. Moreover, the generating functions such as moment, Cumulant, Characteristic, Survival, hazard and Cumulative functions are also derived. The special cases of the H-G family are visualized by shifting the location, scale and shape parameters, two broad and generalized families namely Amoroso (or) Gamma and Beta Prime family of distributions are shown. Under these two families, 56 distributions include Generalized scaled and generalized families of Semi-bounded are exhibited by the authors. Almost all the families of Semi-bounded distributions are special cases of the proposed H-G family and we also discussed the estimation of parameters by the method of unconstrained and constrained maximum likelihood approach by using Non-linear Programming with some applications. Finally, the motivation of the paper is to provide a single distribution which is holistic and generalized nature, in which a semi-bounded random variable has the entirety through the H-G family of distribution.

Some Preliminaries

Explicit expressions for the PDF of H-G family of distribution and the Calculation of constants, generating functions and estimation of the distribution involve several special functions (Prudnikov *et al.* (1986) & Gradshteyn and Ryzhik (2000)) and they are given as follows:

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1. The integral representation of the Kummer's U function is defined by

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty \frac{x^{a-1}e^{-zx}}{(1+x)^{a+1-b}}dx$$

2. The Generalized hyper geometric function of order p and q is defined as

$${}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k(b_2)_k\ldots(b_q)_k}\left(\frac{x^k}{k!}\right)$$

3. The Gauss hyper geometric function is defined as

$${}_2F_1(a_1,a_2;b_1,;x)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(b_1)_k}\left(\frac{x^k}{k!}\right)$$

4. The rising factorial or Pochammer symbol is given as

$$(a)_k=a(a+1)(a+2)\ldots\ldots(a+k-1)$$

5. The integral representation of the Gamma function is defined by

$$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx$$

6. The integral representation of the lower Incomplete Beta function is defined by

$$B(x;a,b)=\int_0^x \frac{t^{a-1}}{(1+t)^{a+b}}dt$$

7. The integral representation of the Di-Gamma function due to Gauss is defined by

$$\Psi(z)=\int_0^\infty\left(\frac{e^{-x}}{x}-\frac{e^{-zx}}{1-e^{-x}}\right)dx$$

8. The Binomial expression of the following series is given by

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$$

9. The Series representation of the Kummer's U function is defined by

$$U(\alpha,\gamma,a/b)=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}{}_1F_1(\alpha,\gamma;a/b)+\frac{(a/b)^{1-\gamma}\Gamma(1-\gamma)}{\Gamma(\alpha)}{}_1F_1(\alpha-\gamma+1,2-\gamma;a/b)$$

$$=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\left(\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\gamma)_k\,k!}(a/b)^k\right)+\frac{(a/b)^{1-\gamma}\Gamma(1-\gamma)}{\Gamma(\alpha)}\left(\sum_{k=0}^{\infty}\frac{(\alpha-\gamma+1)_k}{(2-\gamma)_k\,k!}(a/b)^k\right)$$

10. The first derivative of the Series representation of Kummer's U function with respect 'α' is given as

$$U_{\alpha}^{'}(\alpha,\gamma,a/b)=\left(\begin{array}{l} -\dfrac{\Gamma(1-\gamma)\Psi(\alpha-\gamma+1)}{\Gamma(\alpha-\gamma+1)}\left(\sum_{k=0}^{\infty}\dfrac{(\alpha)_k\left(\Psi(\alpha+k)-\Psi(\alpha)\right)}{(\gamma)_k\,k!}(a/b)^k\right) \\ -\dfrac{(a/b)^{1-\gamma}\Psi(\alpha)\Gamma(1-\gamma)}{\Gamma(\alpha)}\left(\sum_{k=0}^{\infty}\dfrac{(\alpha-\gamma+1)_k\left(\Psi(\alpha-\gamma+1+k)-\Psi(\alpha-\gamma+1)\right)}{(2-\gamma)_k\,k!}(a/b)^k\right)\end{array}\right)$$

11. The first derivative of the Series representation of Kummer's U function with respect'γ' is given as

$$U_{\gamma}^{'}(\alpha,\gamma,a/b)=\left(\begin{array}{l} -\dfrac{\Psi(1-\gamma)\Gamma(1-\gamma)\Psi(\alpha-\gamma+1)}{\Gamma(\alpha-\gamma+1)}\left(\sum_{k=0}^{\infty}\dfrac{(\alpha)_k\left(\Psi(\gamma+k)-\Psi(\gamma)\right)}{(\gamma)_k\,k!}(a/b)^k\right) \\ +\dfrac{(a/b)^{1-\gamma}\Psi(1-\gamma)\Gamma(1-\gamma)}{\Gamma(\alpha)}\left(\sum_{k=0}^{\infty}\dfrac{(\alpha-\gamma+1)_k\left(\Psi(\alpha-\gamma+1+k)-\Psi(\alpha-\gamma+1)\right)(2-\gamma)_k}{\Psi(2-\gamma)k!}(a/b)^k\right)\end{array}\right)$$

12. The first derivative of the Series representation of Kummer's U function with respect to 'a' is given as

$$U_{a}^{'}(\alpha,\gamma,a/b)=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\left(\sum_{k=0}^{\infty}\frac{(\alpha)_k\left(ka^{k-1}\right)}{(\gamma)_k\,k!}(1/b)^k\right)+\frac{\left((1-\gamma)a^{-\gamma}\right)(1/b)^{1-\gamma}\Gamma(1-\gamma)}{\Gamma(\alpha)}\left(\sum_{k=0}^{\infty}\frac{(\alpha-\gamma+1)_k\left(ka^{k-1}\right)}{(2-\gamma)_k\,k!}(1/b)^k\right)$$

13. The first derivative of the Series representation of Kummer's U function with respect to 'b' is given as

$$U_{b}^{'}(\alpha,\gamma,a/b)=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\left(\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\gamma)_k\,k!}(a/b)^k\right)+\frac{(a/b)^{1-\gamma}\Gamma(1-\gamma)}{\Gamma(\alpha)}\left(\sum_{k=0}^{\infty}\frac{(\alpha-\gamma+1)_k}{(2-\gamma)_k\,k!}(a/b)^k\right)$$

1. Introduction to Generalized Distributions

A common problem is that of describing the probability distribution of a single, continuous variable. A few distributions, such as the normal and exponential, were discovered in the 1800's or earlier. But about a century ago the great statistician, Karl Pearson, realized that the known probability distributions were not sufficient to handle all of the phenomena then under investigation, and set out to create new distributions with useful properties. During the 20th century this process continued with abandon and a vast menagerie of distinct mathematical forms were discovered and invented, investigated, analyzed, rediscovered and renamed, all for the purpose of describing the probability of some interesting variable. There are hundreds of named distributions and synonyms in current usage. The apparent diversity is unending and disorienting. Fortunately, the situation is less confused than it might at first appear. Most common, continuous, univariate, unimodal distributions can be organized into a small number of distinct families, which are all special cases of a single Grand Unified

Distribution. The Amoroso, gamma and beta prime family of distributions are considered to be the generalized family of semi-bounded distributions. It includes gamma-exponential, Log-normal, Pearson, Pearson type VII, beta, beta prime, beta exponential, prentice, Pearson IV and many more. The applications of the generalized distributions are extensively studied and the authors shown some of the important and interesting applications are as follows.

Application of Generalized Distributions

Moghaddam *et al.* (2019) argued that a stochastic model of economic exchange, whose steady-state distribution is a Generalized Beta Prime (also known as GB2), and some unique properties of the latter, are the reason for GB2's success in describing wealth/income distributions. They use housing sale prices as a proxy to wealth/income distribution to numerically illustrate this point. The authors also explored parametric limits of the distribution to do so analytically. They discuss parametric properties of the inequality indices – Gini, Hoover, Theil T and Theil L – vis-a-vis those of GB2 and introduce a new inequality index, which serves a similar purpose. They argued that Hoover and Theil L are more appropriate measures for distributions with power-law dependencies, especially fat tails, such as GB2.

Bhatti *et al.* (2019) presented a generalized log Burr III (GLBIII) distribution developed on the basis of a generalized log Pearson differential equation (GLPE). The density function of the GLBIII is exponential, arc, J, reverse-J, bimodal, left-skewed, right-skewed and symmetrical shaped. The hazard rate function of GLBIII distribution has various shapes such as constant, increasing, decreasing, increasing-decreasing, upside-down bathtub and modified bathtub. Descriptive measures such as quantile function, sub-models, ordinary moments, moments of order statistics, incomplete moments, reliability and uncertainty measures are theoretically established. The GLBIII distribution is characterized via different techniques. Parameters of the GLBIII distribution are estimated using maximum likelihood method. A simulation study is performed to illustrate the performance of the maximum likelihood estimates (MLEs). Goodness of fit of this distribution through different methods is studied. The potentiality and usefulness of the GLBIII distribution is demonstrated via its applications to two real data sets.

Oladipo (2019) investigated the polynomials whose coefficients are generalized distribution. Convolution via generalized poly-logarithm and subordination methods were employed to obtain the upper bounds for the first few coefficients of the class defined. Furthermore, relevant connections to Fekete-Szego classical theorem were established, particularly in conic region. Conclusively, consequences of various choices of parameters involved were pointed out. The results further established geometric properties of the generalized distribution associated with univalent functions.

Clementi (2018) proposed the k-generalized distribution as a model for describing the distribution and dispersion of income within a population. Formulas for the shape, moments and standard tools for inequality measurement—such as the Lorenz curve and the Gini coefficient—are given. A method for parameter estimation is also discussed. The model is shown to fit extremely well the data on personal income distribution in Australia and the United States.

Ramoz and louzada (2018) presented a Bayesian reference analysis for the generalized gamma distribution by using a reference prior, which has important properties such as one-to-one invariance under reparameterization, consistent marginalization, consistent sampling and leads to a proper posterior density.

Tripathi *et al.* (2018) introduced a generalized inverse x-gamma distribution (GIXGD) as the generalized version of the inverse x-gamma distribution. The proposed model exhibits the pattern of non-monotone hazard rate and belongs to family of positively skewed models. The explicit expressions of some distributional properties, such as, moments, inverse moments, conditional moments, mean deviation, quantile function have been derived. The maximum likelihood estimation procedure has been used to estimate the unknown model parameters as well as survival characteristics of GIXGD. The practical applicability of the proposed model has been illustrated through a survival data of guinea pigs.

Mansoor *et al.* (2018) introduced a three-parameter extension of the exponential distribution which contains sub-models as the exponential, logistic-exponential and Marshall-Olkin exponential distributions. The new model is very flexible and its associated density function can be decreasing or unimodal. Further, it can produce all of the four major shapes of the hazard rate, that is, increasing, decreasing, bathtub and upside-down bathtub. Given that closed-form expressions are available for the survival and hazard rate functions, the new distribution is quite tractable. It can be used to analyze various types of observations including censored data. Computable representations of the quantile function, ordinary and incomplete moments, generating function and probability density function of order statistics are obtained. The maximum likelihood method is utilized to estimate the model parameters. A simulation study is carried out to assess the performance of the maximum likelihood estimators. Two actual data sets are used to illustrate the applicability of the proposed model.

The Generalized gamma (GG) distribution plays an important role in statistical analysis. For this distribution, **Ramos *et al.* (2017)** derived non-informative priors using formal rules, such as Jeffreys prior, maximal data information prior and reference priors. We have shown that these most popular formal rules with natural ordering of parameters, lead to priors with improper posteriors. This problem is overcome by considering a prior averaging approach discussed in Berger *et al.* [Overall objective priors. Bayesian Analysis. 2015;10(1):189–221]. The obtained hybrid Jeffreys-reference prior is invariant under one-to-one transformations and yields a proper posterior distribution. The authors obtained good frequentist properties of the proposed prior using a detailed simulation study. Finally, an analysis of the maximum annual discharge of the river Rhine at Lobith is presented.

Progri (2016) discussed the exponential generalized Beta distribution (EGBD). For the EGBD model the author provided the closed form expression of the cumulative distribution function (cdf), statistics for special cases and the computation of the mean and variance for the general case. Numerical results are derived for each case to validate the theoretical models presented in the paper. As seen from the equations in the paper, for the computation of the mean requires only two digamma functions and one hypergeometric function; however, the computation of the variance requires the computation of two polygamma functions of the first order, two digamma functions, one hypergeometric function, and two Kampé de Fériet functions.

VedoVatto *et al.* (2016) introduced a new four-parameter model called the Exponentiated Generalized Nadarajah-Haghighi (EGNH) distribution in order to verify this requirement. They proved that its hazard rate function can be constant, decreasing, increasing, upside-down bathtub and bathtub-shape. Theoretical essays are provided about the EGNH shapes. It includes as special models the exponential, exponentiated exponential, Nadarajah-Haghighi's exponential and exponentiated Nadarajah-Haghighi distributions. The authors presented a physical interpretation for the EGNH distribution and obtain some of its mathematical properties including shapes, moments, quantile, generating functions, mean deviations and Renyi entropy. They estimated its parameters by maximum likelihood, on which one of the estimates may be written in closed-form expression. This last result is assessed by means of a Monte Carlo simulation study. The usefulness of the introduced model is illustrated by means of two real data sets. The authors hope that the new distribution could be an alternative to other distributions available for modeling positive real data in many areas.

Merovci (2014) generalized the generalized Rayleigh distribution using the quadratic rank transmutation map studied by Shaw *et al.* to develop a transmuted generalized Rayleigh distribution. The author provided a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The usefulness of the transmuted generalized Rayleigh distribution for modeling data is illustrated using real data.

Potdar and Shirke (2013) introduced a generalized inverted scale family of distributions. Maximum likelihood estimators (MLEs) of scale and shape parameters are obtained. Asymptotic confidence intervals for both the parameters based on the MLE are also constructed. Generalized inverted half-logistic distribution is considered as a member of the generalized inverted scale family. Simulation study is conducted to investigate performance of estimates and confidence intervals for this distribution.

Cordeiro *et al.* (2012) proposed and studied the Kumaraswamy generalized half-normal distribution for modeling skewed positive data. The half-normal and generalized half-normal (Cooray and Ananda, 2008) distributions are special cases of the new model. Several of its structural properties are derived, including explicit expressions for the density function, moments, generating and quantile functions, mean deviations and moments of the order statistics. They investigated maximum likelihood estimation of the parameters and derive the expected information matrix. The proposed model is modified to open the possibility that long-term survivors may be presented in the data.

Nassar and Nada (2011) proposed a new distribution called the beta-generalized Pareto. Several properties of this distribution are presented. The expressions for the mean, mean deviation, variance, and entropies are obtained. The method of maximum likelihood is proposed to estimate the parameters of the distribution. The flexibility of this distribution is illustrated in an application to a real data set.

Elfattah *et al.* (2010) obtained the tables of critical values of modified Kolmogorov-Smirnov (KS) test, Cramer-Von Mises (CVM) test, Anderson-Darling (AD) and Watson test for generalized Frechet distribution with unknown parameters. The sampling distributions for these tests statistics are investigated. Here, they Monte Carlo and Pearson system techniques to create tables of critical values for such situations. Furthermore, we present power comparison between KS test, CVM test, AD test and Watson test.

Scott *et al.* (2009) demonstrated a recursive method for obtaining the moments of the generalized hyperbolic distribution. The method is readily programmable for numerical evaluation of moments. For low order moments we also give an alternative derivation of the moments of the generalized hyperbolic distribution.

The expressions given for these moments may be used to obtain moments for special cases such as the hyperbolic and normal inverse Gaussian distributions. Moments for limiting cases such as the skew hyperbolic t and variance gamma distributions can be found using the same approach.
Based on the above reviews and applications of the generalized distributions, the authors motivated to propose a Holistic and generalized family of distribution which is called as H-G family of distribution with semi-bounded nature of the random variable, where almost all the family of semi-bounded distributions comes under the H-G family. The characteristics and properties of the proposed distribution are extensively studied in the following sections.

H-G Family of Distribution

Definition 1.1: Let *X* be the random variable followed H-G family of distribution with a single location (μ),three scale parameters $(a,b,\sigma)$and three shape parameters (α,β,γ),then it's density function is defined as

$$f_X(x)=|\beta|b^{\alpha}\left(\sigma\,\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\frac{\left(\frac{x-\mu}{\sigma}\right)^{\alpha\beta-1}e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1+b\left(\frac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}} \tag{1.1}$$

Where $\mu\le x<\infty,\mu,\sigma,\gamma,\alpha,\beta,b>0,a\ge 0$ and $\Gamma(\ ),U(\ )$ are the gamma and Kummer's U function respectively.

Theorem 1.2: From(1.1),if $Z=(X-\mu)/\sigma$,then the H-G family of distribution was transformed into Standard H-G family of distribution with two scale parameters $(a,b)$and three shape parameters (α,β,γ),then it's density function is given as

$$f_Z(z)=|\beta|b^{\alpha}\left(\ \Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\frac{z^{\alpha\beta-1}e^{-az^{\beta}}}{\left(1+bz^{\beta}\right)^{\alpha+1-\gamma}} \tag{1.2}$$

Where $0\le x<\infty,\gamma,\alpha,\beta,b>0,a\ge 0$ and $\Gamma(\ ),U(\ )$ are the gamma and Kummer's U function respectively.

Theorem 1.3: The cumulative distribution function of the H-G family of distribution is defined by

$$F_X(x) = \left(\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\sum_{k=0}^{\infty}\left(\frac{(-a/b)^k}{k!}B\left(b\left((x-\mu)/\sigma\right)^{\beta};\alpha+k,1-\gamma-k\right)\right) \tag{1.3}$$

Where $B(\ )$ is the lower Incomplete beta function respectively.

Proof: Let the Cumulative distribution function of a distribution is

$$F_X(x) = \int_{\mu}^{x} f_S(s)\,ds$$

$$F_X(x) = \int_{\mu}^{x}|\beta|b^{\alpha}\left(\sigma\,\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\frac{\left(\frac{s-\mu}{\sigma}\right)^{\alpha\beta-1}e^{-a\left(\frac{s-\mu}{\sigma}\right)^{\beta}}}{\left(1+b\left(\frac{s-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}}\,ds \tag{1.4}$$

By Setting $t = b\left((s-\mu)/\sigma\right)^{\beta}$ in (1.4), then the integral becomes

$$= \left(\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\int_{0}^{b((x-\mu)/\sigma)^{\beta}}\frac{t^{\alpha-1}e^{-(a/b)t}}{(1+t)^{\alpha+1-\gamma}}\,dt \tag{1.5}$$

Similarly, expand the exponent $e^{-(a/b)t} = \sum_{k=0}^{\infty}(-a/b)^k t^k/k!$ in (1.5), and then the final integral expression can be written as

$$F_X(x) = \left(\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\sum_{k=0}^{\infty}\frac{(-a/b)^k}{k!}\int_{0}^{b((x-\mu)/\sigma)^{\beta}}\frac{t^{\alpha+k-1}}{(1+t)^{\alpha+k+1-\gamma-k}}\,dt \tag{1.6}$$

Finally, integrate (1.6), then the final expression of CDF as

$$F_X(x) = \left(\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}\sum_{k=0}^{\infty}\left(\frac{(-a/b)^k}{k!}B\left(b\left((x-\mu)/\sigma\right)^{\beta};\alpha+k,1-\gamma-k\right)\right)$$

2. Constants

Theorem 2.1: The r^{th} moment of the H-G family of distribution is given as

$$E_X\left(x^r\right) = \frac{\mu^r}{\Gamma(\alpha)U(\alpha,\gamma,a/b)}\sum_{k=0}^{r}\binom{r}{k}\left(\sigma/\mu b^{1/\beta}\right)^k\Gamma\left(\alpha+(k/\beta)\right)U\left(\alpha+(k/\beta),\gamma+(k/\beta),a/b\right) \tag{2.1}$$

Proof: The r^{th}order moment of the distribution is

$$E_X\left(x^r\right)=|\beta|b^{\alpha}\left(\sigma\,\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)\right)^{-1}\int_{\mu}^{\infty}x^r\,\frac{\left(\dfrac{x-\mu}{\sigma}\right)^{\alpha\beta-1}e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1+b\left(\dfrac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}}\,dx$$

By Setting $z=\left(X-\mu\right)/\sigma$, then the integral becomes

$$=|\beta|b^{\alpha}\left(\,\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)\right)^{-1}\int_{0}^{\infty}\left(\mu+\sigma z\right)^r\,\frac{z^{\alpha\beta-1}e^{-az^{\beta}}}{\left(1+bz^{\beta}\right)^{\alpha+1-\gamma}}\,dz \tag{2.2}$$

Now expand the term $\left(\mu+\sigma z\right)^r$ by using binomial series, setting $S=bz^{\beta}$ and integrate (2.2) then the r^{th}order moment of the distribution is

$$E_X\left(x^r\right)=\frac{\mu^r}{\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)}\sum_{k=0}^{r}\binom{r}{k}\left(\sigma/\mu b^{1/\beta}\right)^k\Gamma\left(\alpha+\left(k/\beta\right)\right)U\left(\alpha+\left(k/\beta\right),\gamma+\left(k/\beta\right),a/b\right) \tag{2.3}$$

From (2.3), If r =1, 2, 3, 4 then the following order moments are given as

$$E_X\left(x\right)=\frac{\mu}{\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)}\sum_{k=0}^{1}\binom{1}{k}\left(\sigma/\mu b^{1/\beta}\right)^k\Gamma\left(\alpha+\left(k/\beta\right)\right)U\left(\alpha+\left(k/\beta\right),\gamma+\left(k/\beta\right),a/b\right) \tag{2.4}$$

$$E_X\left(x^2\right)=\frac{\mu^2}{\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)}\sum_{k=0}^{2}\binom{2}{k}\left(\sigma/\mu b^{1/\beta}\right)^k\Gamma\left(\alpha+\left(k/\beta\right)\right)U\left(\alpha+\left(k/\beta\right),\gamma+\left(k/\beta\right),a/b\right) \tag{2.5}$$

$$E_X\left(x^3\right)=\frac{\mu^3}{\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)}\sum_{k=0}^{3}\binom{3}{k}\left(\sigma/\mu b^{1/\beta}\right)^k\Gamma\left(\alpha+\left(k/\beta\right)\right)U\left(\alpha+\left(k/\beta\right),\gamma+\left(k/\beta\right),a/b\right) \tag{2.6}$$

$$E_X\left(x^4\right)=\frac{\mu^4}{\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)}\sum_{k=0}^{4}\binom{4}{k}\left(\sigma/\mu b^{1/\beta}\right)^k\Gamma\left(\alpha+\left(k/\beta\right)\right)U\left(\alpha+\left(k/\beta\right),\gamma+\left(k/\beta\right),a/b\right) \tag{2.7}$$

Theorem 2.2: The Skewness and Kurtosis of the H-G family of distribution is given as

$$Skew\left(X\right)=\frac{K_3}{K_2^{3/2}} \tag{2.8}$$

$$Kurtosis\left(X\right)=\frac{K_4}{K_2^2} \tag{2.9}$$

where K_2, K_3, K_4 are the Central moments of the distribution respectively.

Proof: The r^{th} central moments of the distribution are given as

$$K_r = E_X\left(x-E_X\left(x\right)\right)^r \tag{2.10}$$

From (2.10), If r =1, 2, 3, 4 then the following central moments are given as

$$K_1 = E_X\left(x-E_X\left(x\right)\right)=0 \tag{2.11}$$

$$K_2 = E_X\left(x-E_X\left(x\right)\right)^2 = E_X\left(x^2\right)-\left(E_X\left(x\right)\right)^2 \tag{2.12}$$

$$K_3 = E_X\left(x-E_X\left(x\right)\right)^3 = E_X\left(x^3\right)+4\left(E_X\left(x\right)\right)^3+3E_X\left(x^2\right)E_X\left(x\right) \tag{2.13}$$

$$K_4 = E_X\left(x-E_X\left(x\right)\right)^4 = 2E_X\left(x^4\right)+7E\left(x^2\right)-8E\left(x^3\right)E\left(x\right) \tag{2.14}$$

Then substitute the Non-central moments from (2.4), (2.5), (2.6) and (2.7) in (2.11), (2.12), (2.13) and (2.14), we get the exact structure of the Central moments and then Substitute these in (2.8) and (2.9), the skewness and Kurtosis of the H-G family of distribution is found.

Theorem 2.3: The Shannon's differential entropy of the H-G family of distribution is

$$h=\sum_{j=1}^{4}\omega_j\left(\sigma,\alpha,\beta,\gamma,a,b\right) \tag{2.15}$$

Proof: It is found from

$$h=-\int_{\mu}^{\infty} f_X\left(x\right)\log\left(f_X\left(x\right)\right)dx$$

$$=-\int_{\mu}^{\infty}|\beta|b^{\alpha}\left(\sigma\Gamma\left(\alpha\right)U\left(\alpha,\gamma,a/b\right)\right)^{-1}\frac{\left(\frac{x-\mu}{\sigma}\right)^{\alpha\beta-1}e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1+b\left(\frac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}}\log\left(|\beta|b^{\alpha}\left(\sigma\Gamma\left(\alpha\right)U\left(\alpha,\gamma,a/b\right)\right)^{-1}\frac{\left(\frac{x-\mu}{\sigma}\right)^{\alpha\beta-1}e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1+b\left(\frac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}}\right)dx \tag{2.16}$$

By Setting $S=b\left(\left(x-\mu\right)/\sigma\right)^{\beta}$ and integrate (2.16), we will get the final form of Shannon's differential entropy is

$$h=\sum_{j=1}^{4}\omega_j\left(\sigma,\alpha,\beta,\gamma,a,b\right) \tag{2.17}$$

where

$$\omega_1\left(\sigma,\alpha,\beta,\gamma,a,b\right)=-\log\left(\frac{|\beta|b^{\alpha}\sigma\,\Gamma(\alpha)}{U\left(\alpha,\gamma,a/b\right)}\right)$$

$$\omega_2\left(\sigma,\alpha,\beta,\gamma,a,b\right)=\left(\begin{array}{l}-\dfrac{(\alpha\beta-1)\log b}{\beta}\\ +\dfrac{(\alpha\beta-1)\left(\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1}}{\beta}\displaystyle\sum_{k=0}^{\infty}\frac{(-a/b)^k}{k!}\left(\dfrac{\begin{array}{l}\Gamma(a+k)\cot(\pi\gamma)\Gamma(-\gamma-k)\left[\Psi(k+\gamma)+\pi\cot(k\pi)-\Psi(a+k)\right]\\+(k+\gamma)\left[\left(\Psi(k+\gamma)-\cot(k\pi)\Psi(a+\gamma)-\pi\right)\right]\end{array}}{\Gamma(a+1-\gamma)\left(\cot(\pi\gamma)+\cot(k\pi)\right)}\right)\end{array}\right)$$

$$\omega_3\left(\sigma,\alpha,\beta,\gamma,a,b\right)=\frac{a\alpha U\left(\alpha+1,\gamma+1,a/b\right)}{bU\left(\alpha,\gamma,a/b\right)}$$

$$\omega_4\left(\sigma,\alpha,\beta,\gamma,a,b\right)=\left(\alpha+1-\gamma\right)\left(\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)\right)^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\Gamma\left(\alpha+k+1\right)U\left(\alpha+k+1,\gamma+k+1,a/b\right)$$

$\Gamma(\ )$, $\Psi(\ )$ and $U(\ )$ are the Gamma, Di-gamma and Kummer's U functions respectively.

3. Generating Functions

Theorem 3.1: The Moment generating function of H-G family of distribution is given as

$$M_X(t)=\frac{e^{\mu t}}{\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)}\sum_{k=0}^{\infty}\frac{\left(\left(\sigma/b^{1/\beta}\right)t\right)^k}{k!}\Gamma\left(\alpha+(k/\beta)\right)U\left(\alpha+(k/\beta),\gamma+(k/\beta),a/b\right) \tag{3.1}$$

Proof: Let the moment generating function of a distribution is given as

$$M_X(t)=|\beta|b^{\alpha}\left(\sigma\,\Gamma(\alpha)U\left(\alpha,\gamma,a/b\right)\right)^{-1}\int_{\mu}^{\infty}e^{tx}\frac{\left(\dfrac{x-\mu}{\sigma}\right)^{\alpha\beta-1}e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1+b\left(\dfrac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}}dx \tag{3.2}$$

By Setting $z=\left(X-\mu\right)/\sigma$ and expand the exponent in (3.2), then the integral becomes

$$= e^{\mu t} |\beta| b^{\alpha} \left(\Gamma(\alpha) U(\alpha,\gamma,a/b) \right)^{-1} \sum_{k=0}^{\infty} \frac{(\sigma t)^k}{k!} \int_0^{\infty} \frac{z^{\alpha\beta+k-1} e^{-az^{\beta}}}{\left(1+bz^{\beta}\right)^{\alpha+1-\gamma}} dz$$

(3.3)

Rewrite (3.3) by Setting $S = bz^{\beta}$, then integrate it and the final form of MGF is given as

$$M_X(t) = \frac{e^{\mu t}}{\Gamma(\alpha) U(\alpha,\gamma,a/b)} \sum_{k=0}^{\infty} \frac{\left(\left(\sigma / b^{1/\beta}\right) t\right)^k}{k!} \Gamma\left(\alpha + (k/\beta)\right) U\left(\alpha + (k/\beta), \gamma + (k/\beta), a/b\right)$$

Theorem 3.2: The Cumulant of the H-G family of distribution is

$$C_X(t) = \mu t - \log\left(\Gamma(\alpha) U(\alpha,\gamma,a/b)\right) + \log\left(\sum_{k=0}^{\infty} \frac{\left(\left(\sigma / b^{1/\beta}\right) t\right)^k}{k!} \Gamma\left(\alpha + (k/\beta)\right) U\left(\alpha + (k/\beta), \gamma + (k/\beta), a/b\right) \right)$$

(3.4)

Proof: It is found from

$$C_X \; t = \log \; M_X \; t$$

Theorem 3.3: The Characteristic function of the H-G family of distribution is given as

$$\phi_X(t) = \frac{e^{i\mu t}}{\Gamma(\alpha) U(\alpha,\gamma,a/b)} \sum_{k=0}^{\infty} \frac{\left(\sigma i t / b^{1/\beta}\right)^k}{k!} \Gamma\left(\alpha + (k/\beta)\right) U\left(\alpha + (k/\beta), \gamma + (k/\beta), a/b\right)$$

(3.5)

Proof: Let the Characteristic function of a distribution is given as

$$\phi_X(t) = |\beta| b^{\alpha} \left(\sigma \, \Gamma(\alpha) U(\alpha,\gamma,a/b) \right)^{-1} \int_{\mu}^{\infty} e^{itx} \frac{\left(\frac{x-\mu}{\sigma}\right)^{\alpha\beta-1} e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1 + b\left(\frac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}} dx$$

(3.6)

By Setting $z = (X-\mu)/\sigma$ and expand the exponent in (3.6), then the integral becomes

$$= e^{i\mu t} |\beta| b^{\alpha} \left(\Gamma(\alpha) U(\alpha,\gamma,a/b) \right)^{-1} \sum_{k=0}^{\infty} \frac{(i\sigma t)^k}{k!} \int_0^{\infty} \frac{z^{\alpha\beta+k-1} e^{-az^{\beta}}}{\left(1+bz^{\beta}\right)^{\alpha+1-\gamma}} dz$$

(3.7)

Rewrite (3.7) by Substituting $S = bz^{\beta}$, then integrate and the final form of CF is given as

$$\phi_X(t) = \frac{e^{i\mu t}}{\Gamma(\alpha)U(\alpha,\gamma,a/b)} \sum_{k=0}^{\infty} \frac{\left(\sigma it / b^{1/\beta}\right)^k}{k!} \Gamma\left(\alpha + (k/\beta)\right) U\left(\alpha + (k/\beta), \gamma + (k/\beta), a/b\right)$$

Theorem 3.4: The survival function of the H-G family of distribution is

$$S_X(x) = 1 - \left(\Gamma(\alpha)U(\alpha,\gamma,a/b)\right)^{-1} \sum_{k=0}^{\infty} \left(\frac{(-a/b)^k}{k!} B\left(b\left((x-\mu)/\sigma\right)^{\beta}; \alpha+k, 1-\gamma-k\right)\right) \tag{3.8}$$

Where $B(\ )$ is the lower Incomplete Beta function respectively.

Proof: It is found from the following fact

$$S_X(x) = 1 - F_X(x)$$

Theorem 3.5: The hazard function of the H-G family of distribution is

$$h_X(x) = \frac{\left(|\beta| b^{\alpha} \left(\sigma \Gamma(\alpha) U(\alpha,\gamma,a/b)\right)^{-1}\right) \dfrac{\left(\dfrac{x-\mu}{\sigma}\right)^{\alpha\beta-1} e^{-a\left(\frac{x-\mu}{\sigma}\right)^{\beta}}}{\left(1 + b\left(\dfrac{x-\mu}{\sigma}\right)^{\beta}\right)^{\alpha+1-\gamma}}}{1 - \left(\Gamma(\alpha) U(\alpha,\gamma,a/b)\right)^{-1} \sum_{k=0}^{\infty} \left(\dfrac{(-a/b)^k}{k!} B\left(b\left((x-\mu)/\sigma\right)^{\beta}; \alpha+k, 1-\gamma-k\right)\right)} \tag{3.9}$$

Proof: It is found from

$$h_X(x) = \frac{f_X(x)}{S_X(x)} \quad \text{and} \quad S_X(x) = 1 - F_X(x)$$

Theorem 3.6: The Cumulative hazard function of the H-G family of distribution is

$$H_X(x) = -\log\left(1 - \left(\Gamma(\alpha) U(\alpha,\gamma,a/b)\right)^{-1} \sum_{k=0}^{\infty} \left(\frac{(-a/b)^k}{k!} B\left(b\left((x-\mu)/\sigma\right)^{\beta}; \alpha+k, 1-\gamma-k\right)\right)\right) \tag{3.10}$$

Proof*:* Let the Cumulative hazard function of a distribution is given as

$$H_X(x) = -\log\left(1 - F_X(x)\right)$$
$$= -\log\left(\mathrm{S}_X(x)\right)$$

$$H_X\ x = -\log\left(1 - \ \Gamma\ \alpha\ U\ \alpha,\gamma,a/b\ ^{-1}\sum_{k=0}^{\infty}\left(\frac{-a/b^{\ k}}{k!}B\ b\ \ x-\mu\ /\sigma\ ^{\beta};\alpha+k,1-\gamma-k\ \right)\right)$$

4. Special Cases of H-G Family

Result 4.1 : Table-1 shows the two important broad families of H-G family as special cases from (1.1) for different settings of parameters are given.

Table1

Case No	Generalized family	Parameters						
		Location	Scale			Shape		
		μ	a	b	σ	α	β	γ
1	Amoroso & Gamma	μ	1	1	σ	α	β	$1+\alpha$
2	Beta Prime	μ	0	1	σ	α	β	$1-\omega$

Result 4.2 : Table-2 shows the sub and generalized Amoroso and Gamma family of distribution from the H-G family as special cases for different settings of parameters are given.

Table 2

Case No.	Amoroso family	Parameters			
		Location	Scale	Shape	
		μ	σ	α	β
1	Stacy	0	σ	α	β
2	Half exponential power	μ	σ	$1/\beta$	β
3	Generalized Fisher tippet	μ	$\omega/n^{1/\beta}$	n	β
4	Fisher tippet	μ	ω	1	β
5	Generalized Frechet	μ	$\omega/n^{1/\beta}$	n	< 0
6	Frechet	μ	ω	1	< 0

7	Scaled Inverse chi	0	$1/\sqrt{2\omega^2}$	$k/2$	-2
8	Inverse chi	0	$1/\sqrt{2}$	$k/2$	-2
9	Inverse Rayleigh	0	$1/\sqrt{2\omega^2}$	1	-2
10	Pearson Type V	μ	σ	α	-1
11	Inverse Gamma	0	σ	α	-1
12	Scaled Inverse Chi-square	0	$1/2\omega$	$k/2$	-1
13	Inverse Chi-square	0	½	$k/2$	-1
14	Levy	μ	$\omega/2$	½	-1
15	Inverse exponential	0	σ	1	-1
16	Pearson Type III	μ	σ	α	1
17	Gamma	0	σ	α	1
18	Erlang	0	>0	n	1
19	Standard Gamma	0	1	α	1
20	Scaled Chi-square	0	2ω	$k/2$	1
21	Chi-square	0	2	$k/2$	1
22	Exponential	μ	σ	1	1
23	Wien	0	σ	4	1
24	Hohlfeld	0	σ	2/3	3/2
25	Nakagami	μ	σ	α	2
26	Scaled Chi	0	$\sqrt{2\omega}$	$k/2$	2
27	Chi	0	$\sqrt{2}$	$k/2$	2
28	Half normal	0	$\sqrt{2}\omega$	½	2
29	Rayleigh	0	$\sqrt{2}\omega$	1	2
30	Maxwell-Boltzmann	0	$\sqrt{2}\omega$	3/2	2
31	Wilson- Hilferty	0	σ	α	3
32	Generalized Weibull	μ	$\omega/n^{1/\beta}$	n	>0
33	Weibull	μ	ω	1	>0
34	Pseudo-Weibull	μ	ω	$1+1/\beta$	>0

Result 4.3: Table-3 shows the sub families of generalized beta Prime distribution from the H-G family as special cases for different settings of parameters are given.

Table 3

Case No.	Generalized Beta Prime family	Parameters				
		Location	Scale	Shape		
		μ	σ	α	ω	β
1	Burr	μ	σ	1	ω	β
2	Dagum	0	1	α	1	β
3	Paralogistic	0	1	1	β	β
4	Inverse Paralogistic	0	1	β	1	β
5	log-logistic	0	σ	1	1	β
6	Transformed Beta	0	σ	α	ω	β
7	Half Generalized Pearson VII	μ	σ	$1/\beta$	$m-1/\beta$	β
8	Beta Prime	μ	σ	α	ω	1
9	Lomax	μ	σ	1	ω	1
10	Inverse Lomax	μ	σ	α	1	1
11	Standard Beta Prime	0	1	α	ω	1
12	F	0	k_2 / k_1	$k_1 / 2$	$k_2 / 2$	1
13	Uniform Prime	μ	σ	1	1	1

14	Exponential ratio	0	σ	1	1	1
15	Half-Pearson VII	μ	σ	1/2	ω	2
16	Generalized Scaled Half Cauchy	μ	σ	$1/\beta$	$1-(1/\beta)$	β
17	Generalized Half Cauchy	0	1	$1/\beta$	$1-(1/\beta)$	β
18	Scaled Half Cauchy	μ	σ	1/2	1/2	2
19	Generalized scaled Half - t	μ	$\lambda\, v^{1/\beta}$	$1/\beta$	v/β	β
20	Generalized Half-t	0	$v^{1/2}$	$1/\beta$	v/β	β
21	Scaled Half-t	μ	$\lambda\, v^{1/2}$	$1/2$	$v/2$	2
22	Half-t	0	$v^{1/2}$	$1/2$	$v/2$	2

5. Parameter Estimation

Result 5.1: As a first approach**,** we consider the estimation by the method of maximum likelihood. The log-likelihood function for a random sample $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ from (1.1) is

$$\log L(\mu,\sigma,\alpha,\beta,a,b,\gamma)=\begin{pmatrix} n\left(\log|\beta|+\alpha\log b-\log\sigma-\log\Gamma(\alpha)-\log U(\alpha,\gamma,a/b)\right) \\ +(\alpha\beta-1)\sum_{i=1}^{n}\log\left(\frac{x_i-\mu}{\sigma}\right)-a\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}-(\alpha+1-\gamma)\sum_{i=1}^{n}\log\left(1+b\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}\right)\end{pmatrix} \tag{5.1}$$

The first order derivatives of (5.1) with respect to seven parameters are

$$\frac{\partial\log L}{\partial\mu}=-(\alpha\beta-1)\sum_{i=1}^{n}\left(\frac{1}{x_i-\mu}\right)+\frac{a\beta}{\sigma^{\beta}}\sum_{i=1}^{n}(x_i-\mu)^{\beta-1}-b\beta(\alpha+1-\gamma)\sum_{i=1}^{n}\left(\frac{(x_i-\mu)^{\beta-1}}{\sigma^{\beta}+b(x_i-\mu)^{\beta}}\right) \tag{5.2}$$

$$\frac{\partial\log L}{\partial\sigma}=-\frac{n}{\sigma}-(\alpha\beta-1)\frac{n}{\sigma}+\frac{a\beta}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}+\frac{(\alpha+1-\gamma)b\beta}{\sigma^{\beta+1}}\sum_{i=1}^{n}\left(\frac{(x_i-\mu)^{\beta}}{\sigma^{\beta}+b(x_i-\mu)^{\beta}}\right) \tag{5.3}$$

$$\frac{\partial \log L}{\partial \alpha}=n\left(\log b-\Psi(\alpha)-\frac{U_{\alpha}^{'}(\alpha,\gamma,a/b)}{U(\alpha,\gamma,a/b)}\right)+\beta\sum_{i=1}^{n}\log\left(\frac{x_i-\mu}{\sigma}\right)-\sum_{i=1}^{n}\log\left(1+b\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}\right) \tag{5.4}$$

$$\frac{\partial \log L}{\partial \beta}=\frac{n\beta}{|\beta|^2}+\alpha\sum_{i=1}^{n}\log\left(\frac{x_i-\mu}{\sigma}\right)+\frac{a\beta}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}+\frac{(\alpha+1-\gamma)b\beta}{\sigma^{\beta+1}}\sum_{i=1}^{n}\left(\frac{(x_i-\mu)^{\beta}}{\sigma^{\beta}+b(x_i-\mu)^{\beta}}\right) \tag{5.5}$$

$$\frac{\partial \log L}{\partial a}=-\frac{nU_{a}^{'}(\alpha,\gamma,a/b)}{U(\alpha,\gamma,a/b)}-\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\beta} \tag{5.6}$$

$$\frac{\partial \log L}{\partial b}=n\left(\frac{\alpha}{b}-\frac{U_{b}^{'}(\alpha,\gamma,a/b)}{U(\alpha,\gamma,a/b)}\right)-(\alpha+1-\gamma)\sum_{i=1}^{n}\left(\frac{(x_i-\mu)^{\beta}}{\sigma^{\beta}+b(x_i-\mu)^{\beta}}\right) \tag{5.7}$$

and

$$\frac{\partial \log L}{\partial \gamma}=-n\frac{U_{\gamma}^{'}(\alpha,\gamma,a/b)}{U(\alpha,\gamma,a/b)}+\sum_{i=1}^{n}\log\left(1+b\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}\right) \tag{5.8}$$

Setting these expressions to zero and solving them simultaneously yields the maximum likelihood estimates of the seven parameters.

Result 5.2: As a second approach, the authors realized that the computational complexity of the maximum likelihood estimation of the proposed distribution is Painstaking due to the non-linearity of parameters and the involvement of special functions. Hence, they moved to the Non-linear programming approach by adopting Constrained Maximum likelihood method to estimate the parameters of the H-G family of distribution and the idea is to Maximizing the log-likelihood function from (5.1) under some restriction and parameter constraints and it given as follows:

$$Max\left(\log L(\mu,\sigma,\alpha,\beta,a,b,\gamma)\right)=\left(\begin{array}{l} n\left(\log|\beta|+\alpha\log b-\log\sigma-\log\Gamma(\alpha)-\log U(\alpha,\gamma,a/b)\right) \\ +(\alpha\beta-1)\sum_{i=1}^{n}\log\left(\frac{x_i-\mu}{\sigma}\right)-a\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}-(\alpha+1-\gamma)\sum_{i=1}^{n}\log\left(1+b\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}\right)\end{array}\right) \tag{5.9}$$

Subject to the constraints

$$\left(\begin{array}{l} n\left(\log|\beta|+\alpha\log b-\log\sigma-\log\Gamma(\alpha)-\log U(\alpha,\gamma,a/b)\right) \\ +(\alpha\beta-1)\sum_{i=1}^{n}\log\left(\frac{x_i-\mu}{\sigma}\right)-a\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}-(\alpha+1-\gamma)\sum_{i=1}^{n}\log\left(1+b\left(\frac{x_i-\mu}{\sigma}\right)^{\beta}\right)\end{array}\right)\le 0$$

$$\mu,\sigma,\alpha,a,b,\gamma\ge 0$$

$$-\infty<\beta<+\infty$$

6. Application

In this Section, we now illustrate an application of the H-G family of distribution for the 4 variables namely Sepal length (Sepal length of Iris Setosa, Versicolour, Virginica in cms), Plasma glucose (Plasma glucose concentration a 2 hours in an oral glucose tolerance test), Nitric Oxides(nitric oxides concentration (parts per 10 million)) and Age-at-attack(age in years when heart attack occurred) with a random sample size of (*n=30*).The variables are collected from the Databases such as **Iris Plants data(1936), Pima Indians Diabetes(1988), Boston Housing data(1993) and Echocardiogram data (1988)** respectively. The information regarding the databases are clearly given in the references. The authors realized the computational difficulties in the classical unconstrained Maximum likelihood method and hence they adopted Non-linear programming approach to estimate the parameters by using the constrained Maximum likelihood method with the help of Optimization Module in the best known Mathematical Software Maple version 18 and the estimated results of Parameters with a 5 digit decimal approximation are tabulated in Table-4 and the fitted pdf of H-G family of distribution are visualized as follows:

Table-4: Parameter estimates of the H-G family of distribution.

No.	Variable	**Constrained Maximum likelihood estimators of parameters**							
		Location	Scale			Shapes			Maximized Log-
		μ	$\hat{a}$	$\hat{b}$	σ	α	β	$\hat{\gamma}$	$\text{Max}(\log L)$
1	**Sepal Length**	1.03774	0.83608	1.05925	1.10082	1.22388	0.90946	0.89539	**-0.001016**
2	**Plasma Glucose**	1.00289	0.28393	1.27425	1.16024	1.89387	0.24374	0.55255	**-0.002678**
3	**Nitric Oxide**	1.10086	0.88457	1.06468	1.04751	1	1	0.97722	**-0.000188**
4	**Age at Heart attack**	1.00552	0.44988	1.19307	1.17515	1.63858	0.37745	0.68032	**-0.002499**

7. Discussion and Conclusion

In this paper, the authors introduced a new seven parameter Kummer's holistic and generalized family of distribution with semi-bounded which is called as H-G family of distribution. The Characteristics of the distribution was studied in detail and interesting aspect of the proposed distribution due to its special cases. So far in the literature, the grand unified distribution (GUD) is considered to be the generalized family of distribution which all the existing distributions come under the special cases of GUD. But the proposed H-G family is a synonymous family where the two broad and generalized families namely Amoroso and gamma, beta prime are coming under the proposed distribution with semi-bounded nature. By changing the location, scale and shape parameters of the H-G distribution, almost all the generalized and scaled versions of the semi-bounded distributions come under the H-G family. Moreover, the use of Jacobean transformations of standard H-G random variable such as inverse, logarithmic, exponential, transcendental and trigonometric transformations helps the standard H-G family to evolves more generalized family of distributions includes the Finite family. Finally, the authors exhibited the parameter estimation by Constrained Maximum likelihood method and the authors left the computational complexity of parameters in the proposed distribution by using classical maximum likelihood method for future research.

References

Abd-Elfattah, A. M., Fergany, H. A., and Omima, A. M. (2010). Goodness of fit tests for generalized Fréchet distribution. *Australian Journal of Basic and Applied Sciences*, **4(2)**, 286-301.

Bhatti, F. A., Ali, A., Hamedani, G. G., and Ahmad, M. (2019). On the generalized log Burr III distribution: development, properties, characterizations and applications. *Pakistan Journal of Statistics*, **35(1)**.

Clementi, F., Di Matteo, T., Gallegati, M., and Kaniadakis, G. (2008). The κ-generalized distribution: A new descriptive model for the size distribution of incomes. *Physica A: Statistical Mechanics and its Applications*, **387(13)**, 3201-3208.

Cordeiro, G. M., Pescim, R. R., and Ortega, E. M. (2012). The Kumaraswamy generalized half-normal distribution for skewed positive data. *Journal of Data Science*, **10(2),** 195-224.

Crooks, G. E. (2017). Field Guide to Continuous Probability Distributions. *Obtidoem: http://threeplusone. com/FieldGuide. pdf*.

Gradshteyn, I. S. and Ryzhik, I. M. (2000). *Table of Integrals, Series, and Products*, 6th ed. SanDiego: Academic Press. MR1773820

Johnson, N.L. and Kotz S. (1971) Continuous Univariate Distributions, Wiley Series in Probability and Statistics.

Mansoor, M., Tahir, M. H., Cordeiro, G. M., Provost, S. B., and Alzaatreh, A. (2019). The Marshall-Olkin logistic-exponential distribution. *Communications in Statistics-Theory and Methods*, **48(2)**, 220-234.

Merovci, F. (2014). Transmuted generalized Rayleigh distribution. *Journal of Statistics Applications & Probability*, **3(1)**, 9-20.

Moghaddam, M. D., Mills, J., and Serota, R. A. (2019). Generalized Beta Prime Distribution: Stochastic Model of Economic Exchange and Properties of Inequality Indices. *arXiv preprint arXiv:1906.04822*.

Nassar, M. M., and Nada, N. K. (2011). The beta generalized Pareto distribution. *Journal of Statistics: Advances in Theory and Applications*, **6(1/2)**, 1-17.

Oladipo, A. T. (2019). Bounds for Probabilities of the Generalized Distribution Defined by Generalized Polylogarithm. *Journal of Mathematics (ISSN 1016-2526)*, **51(7)**, 19-26.

Potdar, K. G., and Shirke, D. T. (2013). Inference for the parameters of generalized inverted family of distributions. In *ProbStat Forum* (Vol. 6, pp. 18-28).

Progri, I. (2016). Exponential generalized Beta distribution. *J. Geol. Geoinfo. Geointel*, *2016*.

Prudnikov, A. P., Brychkov, Y. A. and Marichev, O. I. (1986). *Integrals and Series*, Vols. 1, 2 and 3. Amsterdam: Gordon and Breach Science Publishers. MR0874986

Ramos, P. L., and Louzada, F. (2018). Bayesian reference analysis for the Generalized Gamma distribution. *IEEE Communications Letters*, **22(9)**, 1950-1953.

Ramos, P. L., Achcar, J. A., Moala, F. A., Ramos, E., and Louzada, F. (2017). Bayesian analysis of the generalized gamma distribution using non-informative priors. *Statistics*, **51(4),** 824-843.

Smith, J.W., Everhart, JE., Dickson, W.C., Knowler, W.C., and Johannes, R.S. (1988). Using the ADAP learning algorithm to forecast the onset of diabetes mellitus. Proceedings of the Symposium on Computer Applications and Medical Care} (pp. 261--265). IEEE Computer Society Press.

Scott, D. J., Würtz, D., Dong, C., and Tran, T. T. (2011). Moments of the generalized hyperbolic distribution. *Computational statistics*, **26(3)**, 459-476.

Salzberg, S. (1988). Exemplar-based learning: Theory and implementation (Technical Report TR-10-88). Harvard University, Center for Research in Computing Technology, Aiken Computation Laboratory (33 Oxford Street; Cambridge, MA 02138)

Tripathi, H., Yadav, A. S., Saha, M., and Kumar, S. (2018). Generalized inverse xgamma distribution: A non-monotone hazard rate model. *arXiv preprint arXiv:1812.04933*.

VedoVatto, T., Nascimento, A. D. C., WR Filho, M., Lima, M. C. S., Pinho, L. G. B., and Cordeiro, G. M. (2016). Some computational and theoretical aspects of the exponentiated generalized Nadarajah-Haghighi distribution. *arXiv preprint arXiv:1610.08876*.

Quinlan, R. (1993). Combining Instance-Based and Model-Based Learning. In Proceedings on the Tenth International Conference of Machine Learning, 236-243, University of Massachusetts, Amherst. Morgan Kaufmann.

Probability curve of fitted *pdf* of H-G family of distribution

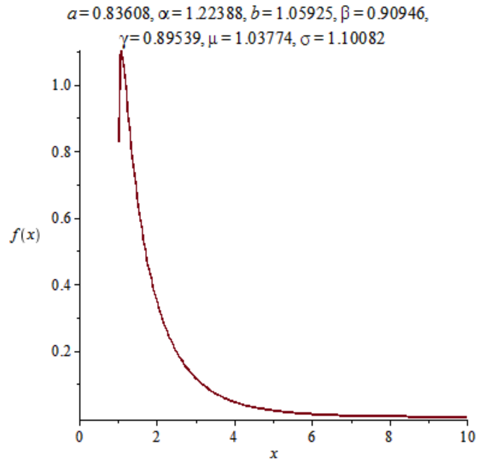

Fig-1(Sepal Length)

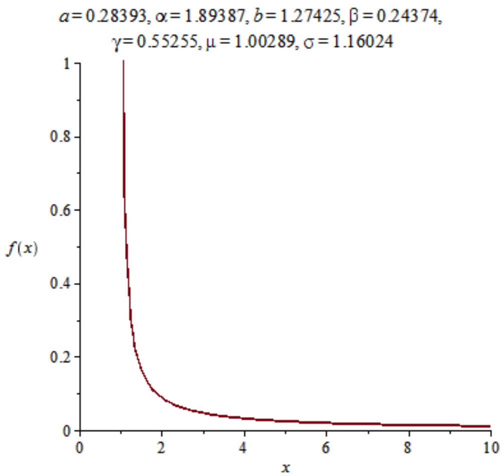

Fig-2 (Plasma Glucose)

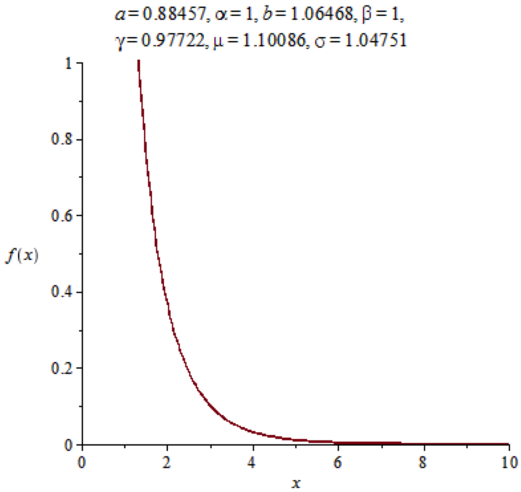

Fig-3 (Nitric Oxide)

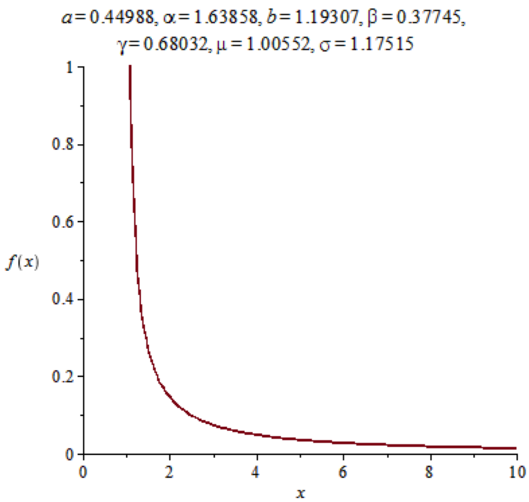

Fig-4(Age at Heart attack)