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Catalogues of Efficient Circular Weakly Balanced Repeated Measurements Designs in Periods of Two Different Sizes

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ABSTRACT

Minimal balanced repeated measurements designs (RMDs) are economical to balance out the residual effects. In the situations where these designs cannot be constructed, weakly balanced RMDs are preferred. In this article, catalogues of circular weakly balanced RMDs are presented in periods of two different sizes which are highly efficient according to the efficiency criteria for (i) residual effects, and (ii) Separability. These catalogues are useful for the practitioners and researchers.

1. Introduction

Repeated measurements designs (RMDs) are widely used in almost every field of life. These designs are economical but with the use of RMDs residual effects may arise. Minimal BRMDs (balanced RMDs) balance out these residual effects at low cost. Weakly balanced RMDs (WBRMDs) are preferred in the situations where minimal BRMDs cannot be constructed. RMD is balanced with respect to the first-order residual effects if each treatment is immediately preceded λ' times by each other treatment (excluding itself). RMD for v (treatments) and p_i (period sizes) is weakly balanced with respect to the first-order residual effects if each treatment is immediately preceded λ'_1 or λ'_2 times by each other treatment (excluding itself) where $|\lambda'_2 - \lambda'_1| = 1$. Williams (1949) initiated for RMDs. Cheng and Wu (1980) constructed balanced and strongly balanced uniform RMDs. Afsarinejad (1994) constructed minimal balanced and strongly balanced RMDs in periods of unequal sizes. Using method of cyclic shifts, Iqbal and Jones (1994), Iqbal and Tahir (2009), Iqbal *et al*. (2010), Rajab *et al*. (2018), Rasheed *et al*. (2018) and Bashir *et al*. (2018) constructed minimal circular balanced and

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strongly balanced RMDs for different cases. Ahmed *et al*. (2018) presented the catalogue of CBRMDs in periods of two different sizes. Some more references of the designs constructed through method of cyclic shifts can be found in Ahmed *et al*. (2020).

Bailey *et al*. (2017) constructed universally optimal WBRMDs for $p = v$. Khan *et al*. (2019) developed some generators to obtain CWBRMDs for $p \leq v$ in periods of equal sizes. Hussain *et al*. (2020) developed some generators to obtain minimal CWBRMDs in periods of two different sizes when p_1 is even. They have not constructed these designs for p_1 odd. In this article, catalogues of minimal CWBRMDs are presented along with their efficiencies for residual effects and for Separability in periods of two different sizes where p_1 (odd) $>p_2$. These designs are constructed through method of cyclic shifts (Rule I).

2. Efficiency of the Proposed Designs

2.1 Efficiency of Residual Effects

In order to consider the efficiency of the constructed designs, the model for circular RMDs proposed by Davis and Hall (1969) is used.

$$\mathbf{Y} = \mu\mathbf{E} + \mathbf{D}\boldsymbol{\delta} + \mathbf{R}\boldsymbol{\rho} + \mathbf{U}\boldsymbol{\nu} + \mathbf{P}\boldsymbol{\pi} + \boldsymbol{\mathcal{E}} \qquad (2.1)$$

Here $\mathbf{Y}$ is the $np\times1$ column vector of the np observations, μ is the overall mean, $\boldsymbol{\delta}$ is the vector of direct effects of order $v\times1$, $\boldsymbol{\rho}$ is residual effect vector of order $v\times1$, $\boldsymbol{\nu}$ is the unit vector of order $n\times1$, $\boldsymbol{\pi}$ is the vector of period effects having order $p\times1$ and $\boldsymbol{\mathcal{E}}$ is random error vector of order $np\times1$ with mean zero and constant variance σ^2. $\mathbf{E}$ is the matrix of 1's with order $p\times q$. **D, R, U, P** are design matrices of observations versus direct effects, residual effects, unit effects and period effects of treatments with order $np\times v$,$np\times v$, $np\times bv$ and $np\times p$ respectively.

Using the identities, $\mathbf{D'D} = \mathbf{R'D} = bp\mathbf{I}_v$, $\mathbf{D'R} = \mathbf{L}$, $\mathbf{D'U} = \mathbf{N}$, $\mathbf{D'P} = b\mathbf{E}_{v,p}$, $\mathbf{R'U} = \mathbf{N}$, $\mathbf{U'U} = p\mathbf{I}_n$, $\mathbf{U'U} = \mathbf{E}_{k,q}$,$\mathbf{P'P}= n\mathbf{I}_n$. The reduced normal equations for $\widehat{\boldsymbol{\delta}}$ and $\widehat{\boldsymbol{\rho}}$ will be:

$$\mathbf{C}\begin{bmatrix}\widehat{\boldsymbol{\delta}}\\ \widehat{\boldsymbol{\rho}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{\pi}\\ \boldsymbol{\pi}' & \boldsymbol{\Theta}\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{\delta}}\\ \widehat{\boldsymbol{\rho}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{S}\end{bmatrix}$$

$$\begin{bmatrix}bp\mathbf{I}_v - p^{-1}\mathbf{N}\mathbf{N}' & \mathbf{L}' - p^{-1}\mathbf{N}\mathbf{N}'\\ \mathbf{L}' - p^{-1}\mathbf{N}\mathbf{N}' & bp\mathbf{I}_v - p^{-1}\mathbf{N}\mathbf{N}'\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{\delta}}\\ \widehat{\boldsymbol{\rho}}\end{bmatrix} = \begin{bmatrix}\mathbf{D}'\mathbf{Y} - p^{-1}\mathbf{N}\mathbf{U}'\mathbf{Y}\\ \mathbf{R}'\mathbf{Y} - p^{-1}\mathbf{N}\mathbf{U}'\mathbf{Y}\end{bmatrix}$$

$\boldsymbol{\theta} = bp\mathbf{I}_{\boldsymbol{v}} - p^{-1}\mathbf{N}\mathbf{N}'$, $\boldsymbol{\Theta} = bp\mathbf{I}_{\boldsymbol{v}} - p^{-1}\mathbf{N}\mathbf{N}'$, $\boldsymbol{\pi} = \mathbf{L}' - p^{-1}\mathbf{N}\mathbf{N}'$,

$\mathbf{T} = \mathbf{D}'\mathbf{Y} - p^{-1}\mathbf{N}\mathbf{U}'\mathbf{Y}$, $\boldsymbol{S} = \mathbf{R}'\mathbf{Y} - p^{-1}\mathbf{N}\mathbf{U}'\mathbf{Y}$,

For the period of two different sizes information matrix can be presented as:

$$\mathbf{C}* = \begin{bmatrix} bp\mathbf{I}_v - {p_1}^{-1}\mathbf{N_1}\mathbf{N'}_1 - {p_2}^{-1}\mathbf{N_2}\mathbf{N'}_2 & \mathbf{L}' - {p_1}^{-1}\mathbf{N_1}\mathbf{N'}_1 - {p_2}^{-1}\mathbf{N_2}\mathbf{N'}_2 \\ \mathbf{L}' - {p_1}^{-1}\mathbf{N_1}\mathbf{N'}_1 - {p_2}^{-1}\mathbf{N_2}\mathbf{N'}_2 & bp\mathbf{I}_v - {p_1}^{-1}\mathbf{N_1}\mathbf{N'}_1 - {p_2}^{-1}\mathbf{N_2}\mathbf{N'}_2 \end{bmatrix}$$

The information matrix for direct and residual effects denoted by $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ respectively can be specified by their initial rows:

$$\boldsymbol{\theta} = [\theta_0, \theta_1, \dots, \theta_{t-1}] \quad \text{and} \quad \boldsymbol{\Theta} = [\Theta_0, \Theta_1, \dots, \Theta_{t-1}]$$

According to the duality presented in the model (2.1), both direct and residual effects share the same information matrix. The non-zero Eigen values of information matrix C^* are called the canonical efficiency factors, see James and Wilkinson (1971) and Pearce *et al*. (1974). The canonical efficiency factor is calculated by working out harmonic mean of non-zero Eigen values of their respective information matrix relative to that of an orthogonal with the same number of treatments having same number of replications. It is further assume that σ^2 is the same for the proposed design and the orthogonal design to which it is compared. The high value of E_r shows that design is suitable for the estimation of residual effects.

2.2 Efficiency of Separability

RMD must be characterized for its ability of separating the direct and first order residual effects. Divecha and Gondaliya (2014) used the following measure of separability called efficiency of separabiliy (Es) for balanced RMDs.

$$\text{Es} = \left[1 - \left\{\frac{(\lambda_3 - \lambda_2)^2}{\lambda_3 + (v-1)\lambda_2(\lambda_1 + \lambda_3 + (v-1)\lambda_2}\right\}^{1/2}\right] \times 100\ \%$$

Where

- λ_1 is number of units allocated for each treatment in two successive periods.
- Each ordered pair of distinct treatments is given to λ_2 units.
- Each pair of treatments with itself is given to λ_3 units.

3. Method of Cyclic Shifts

Method of cyclic shifts introduced by Iqbal (1991) is explained here in detail.

- Method of cyclic shifts (Rule I) is explained here for the construction of CBRMDs and CWBRMDs.

Rule I: Let $S_j = [q_{j1}, q_{j2}, \ldots, q_{j(p_1-1)}]$ and $S_i = [q_{i1}, q_{i2}, \ldots, q_{i(p_2-1)}]$, j = 1, 2, …, i-1 be i sets of shifts, where $1 \leq q_{ij} \leq v$-1. If each element 1, 2, …, v-1 appears an equal number of times, say λ' in a new set of shifts S*, where S* = $[\{q_{j1}, q_{j2}, \ldots, q_{j(p_1-1)}, v\text{-}(q_{j1}+ q_{j2}+\ldots+ q_{j(p_1-1)}) \bmod v\},\{q_{i1}, q_{i2}, \ldots, q_{i(p_2-1)}, v\text{-}(q_{i1}+ q_{i2}+ \ldots+ q_{i(p_2-1)}) \bmod v\}]$ then it will be CBRMD in periods of sizes p_1 and p_2. If S* contains all of 1, 2, …,v-1 exactly λ'_1 time except a few of these which appear λ'_1-1or λ'_1+1times then it will be CWBRMD. If $\lambda'_1 = 1$ then it will be minimal CWBRMD. Sum of any two, three, …, (p-1) consecutive elements of a set should not be 0 (mod v). If so, reorder the elements of the corresponding set for better efficiency.

Example 3.1: Minimal CWBRMD is constructed through the following sets of shifts for v = 10, p_1 = 5 and p_2 = 3.

$S_1 = [3, 4, 8, 6]$ $S_2 = [1, 2]$

Proof: S_1*= [3,4,8,6,10-(3+4+8+6) mod 10 = 9], S_2*= [1,2,10-(1+2) mod 10 = 7]. Combining these two, we get S* = [3,4,8,6,9,1,2,7]. Here each of 1, 2, …, 9 appears exactly once except 5 which does not appear, hence given sets of shifts provide minimal CWBRMD for v = 10, p_1 = 5 and p_2 = 3. Required CWBRMD will be obtained using 20 experimental subjects in the following manner.

Take v experimental subjects for one set of shifts S_1 = [3,4,8,6]. Assign 0, 1, …, v-1 to each subject in first period respectively. To get the elements of second period for each subject, add 3 (mod 10) to the each element of first period for all subjects. Then add 4 (mod 10) to the each element of second period for all subjects of third period. Similarly add 8 and 6 to complete the design.

Periods	**Subjects**									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	0	1	2
3	7	8	9	0	1	2	3	4	5	6
4	5	6	7	8	9	0	1	2	3	4
5	1	2	3	4	5	6	7	8	9	0

Take v more experimental subjects for second set of shifts $S_2 = [1, 2]$. Assign 0, 1, …, 9 to each subject in first period respectively. To get the elements of second period for each subject, add 1 (mod 10) to the each element of first period for all subjects. Then add 2 (mod 10) to the each element of second period for all subjects of third period.

Periods	**Subjects**									
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	0	1	2	3	4	5	6	7	8	9
2	1	2	3	4	5	6	7	8	9	0
3	3	4	5	6	7	8	9	0	1	2

Hence it is required CWBRMD for $v = 10$, $p_1 = 5$ and $p_2 = 3$ in 20 experimental subjects.

3. Catalogues of efficient CWBRMDs in Periods of Two Different Sizes in Which Some Ordered Pairs Do Not Appear as Preceded values

In this Section, catalogues of efficient CWBRMDs are presented in periods of two different sizes. In these designs ordered pairs {(0, v/2), (1, (v+2)/2), …, ((v-2)/2, v-1), (v/2, 0), ((v+2)/2, 1), …, (v-1, (v-2)/2)} do not appear together while all other appear once. These designs are constructed using the following sets of shifts.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $\quad j = 1, 2, \ldots, i.$

$S_{i+1} = [q_{i1}, q_{i2}, \ldots, q_{i(s-1)}]$

Where

- $1 \leq q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{j(s-1)} \leq v\text{-}1$ but $\neq v/2$.
- S* contains each of 1, 2, …, v-1 exactly once except v/2 which does not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v\text{-}(q_{j1}+q_{j2}+\ldots+q_{j(r-1)}), q_{i1}, q_{i2}, \ldots, q_{j(s-1)}, v\text{-}(q_{i1}+q_{i2}+\ldots+q_{i(s-1)})]$

4.1 Catalogue of efficient CWBRMDs when p_1 and p_2 both are odd

Catalogue of efficient CWBRMDs for $v = ri+s+2$, i odd, $p_1 = r$ (odd) and $p_2 = s$ (odd) using (i+1) sets of shifts. Here, $v < 100$, $5 \leq p_1 \leq 9$ and $3 \leq p_2 \leq 7$ is presented in Table 1.

Table 1: Efficient CWBRMDs for $v< 100$, $5 \leq p_1$ (odd) ≤ 9 and $3 \leq p_2$ (odd) ≤ 7.

v	p_1	p_2	**Sets of Shifts**	**Es**	**Er**
10	5	3	[4,3,6,8]+[1,2]	0.84	0.81
20	5	3	[4,8,15,16]+[5,6,12,13]+[1,2,7,9]+[3,18]	0.87	0.81
30	5	3	[2,4,27,28]+[5,10,24,25]+[11,13,21,22]+ [16,17,18,19]+[1,3,6,8]+[7,9]	0.95	0.81
40	5	3	[2,4,37,38]+[8,7,34,35]+[11,13,31,32]+ [23,10,28,29]+[18,24,25,26]+[3,19,15,21]+ [1,5,6,12]+[9,14]	0.96	0.81
50	5	3	[2,4,47,48]+[7,8,44,45]+[11,13,41,42]+ [16,17,38,39]+[20,22,35,36]+[21,30,32,33]+ [3,9,28,29]+[5,24,18,26]+[1,6,10,14]+[12,15]	0.97	0.81
60	5	3	[2,4,57,58]+[7,8,54,55]+[11,13,51,52]+ [16,17,48,49]+[20,22,45,46]+[10,41,42,43]+ [26,37,38,39]+[1,14,34,35]+[3,21,31,32]+ [11,25,27,28]+[5,6,9,15]+[18,19]	0.96	0.80
70	5	3	[2,4,67,68]+[7,8,64,65]+[11,13,61,62]+ [16,17,58,59]+[20,22,55,56]+[10,41,52,53]+ [12,48,49,50]+[28,44,45,46]+[6,9,40,42]+ [5,21,37,38]+[14,23,34,33]+[18,29,30,31]+ [1,3,15,24]+[19,25]	0.95	0.80
80	5	3	[2,4,77,78]+[7,8,74,75]+[11,13,71,72]+ [16,17,68,69]+[20,22,65,66]+[25,26,62,63]+ [3,57,59,60]+[18,53,55,56]+[33,50,51,52]+ [1,15,47,48]+[6,19,44,45]+[12,24,39,42]+ [14,35,36,37]+[23,30,32,34]+[5,9,10,27]+[21,28]	0.93	0.80
90	5	3	[2,4,87,88]+[7,8,84,85]+[11,13,81,82]+ [16,17,78,79]+[20,22,75,76]+[25,26,72,73]+ [3,57,69,70]+[5,64,66,67]+[19,61,62,63]+ [37,56,58,59]+[9,18,50,51]+[34,36,1,54]+ [27,30,31,39]+[12,24,47,48]+[10,38,44,42]+ [21,35,40,41]+[6,14,23,28]+[15,32]	0.97	0.80
12	7	3	[7,1,2,8,9,10]+[3,4]	0.87	0.87
26	7	3	[7,8,21,22,23,24]+[5,9,16,17,18,19]+ [1,2,4,6,10,14]+[3,11]	0.94	0.87
40	7	3	[7,8,35,36,37,38]+[4,12,14,31,32,33]+ [5,15,26,27,28,29]+[2,19,9,21,22,23]+ [1,3,10,11,13,17]+[6,16]	0.96	0.86
54	7	3	[7,8,49,50,51,52]+[12,28,44,45,46,47]+ [5,6,39,40,41,42]+[17,19,36,34,35,37]+ [2,15,20,29,31,32]+[13,21,23,24,25,26]+ [1,3,4,9,10,11]+[14,18]	0.97	0.82

68	7	3	[7,8,63,64,65,66]+[1,9,20,59,60,61]+ [15,45,54,55,56,57]+[10,12,48,49,50,51]+ [26,24,42,43,44,46]+[3,6,37,38,39,40]+ [14,23,33,32,31,35]+[19,18,27,28,29,30]+ [2,4,5,11,13,16]+[21,22]	0.98	0.86
82	7	3	[7,8,77,78,79,80]+[10,30,72,73,74,75]+ [32,33,67,68,69,70]+[3,5,62,63,64,65]+ [16,17,57,58,59,60]+[13,45,52,53,54,55]+ [37,46,47,48,49,50]+[15,40,39,23,42,43]+ [1,11,9,34,35,36]+[2,4,6,12,18,19]+ [14,20,22,24,26,27]+[25,28]	0.98	0.86
96	7	3	[7,8,91,92,93,94]+[10,30,86,87,88,89]+ [20,45,81,82,83,84]+[15,75,76,77,78,79]+ [2,22,70,71,72,73]+[17,32,65,66,67,68]+ [16,58,60,61,62,63]+[11,19,31,55,56,57]+ [14,47,25,49,50,51]+[1,9,13,40,42,43]+ [3,4,35,36,37,38]+[18,23,27,29,28,33]+ [5,6,12,21,41,53]+[24,26]	0.99	0.86
14	9	3	[2,6,4,8,5,12,9,11]+[1,3]	0.89	0.89
32	9	3	[13,15,25,26,27,28,29,30]+ [7,6,18,19,20,21,22,23]+[2,1,3,4,8,5,10,14]+[9,11]	0.95	0.89
50	9	3	[18,10,43,44,45,46,47,48]+ [12,36,15,38,37,39,40,41]+ [6,20,29,30,31,32,33,34]+ [1,2,3,23,16,24,27,26]+ [5,10,12,20,24,31,32,33]+[9,19]	0.91	0.89
68	9	3	[36,60,61,62,63,64,65,66]+ [7,9,53,54,55,56,57,58]+ [30,35,46,47,48,49,50,51]+ [21,25,39,40,41,42,43,44]+ [5,10,12,20,24,31,32,33]+ [14,17,18,22,19,23,27,26]+ [1,2,3,4,6,8,13,15]+[11,28]	0.98	0.89
86	9	3	[18,10,79,80,81,82,83,84]+ [30,47,72,73,74,75,76,77]+ [21,19,65,66,67,68,69,70]+ [1,2,58,59,60,61,62,63]+ [35,20,51,52,53,54,55,56]+ [9,42,11,44,45,46,48,49]+ [7,8,12,36,37,38,39,40]+[3,4,5,17,23,27,28,32]+ [6,13,14,15,16,24,222,25]+[26,29]	0.97	0.89
14	7	5	[3,5,12,6,9,8]+[1,4,2,10]	0.89	0.90
28	7	5	[7,8,23,24,25,26]+[3,9,18,19,20,21] + [5,12,6,13,16, 15]+[1,2,4,10]	0.95	0.87

42	7	5	[7,8,37,38,39,40]+[18,22,32,33,34,35]+ [12,11,27,28,30,29]+[3,5,20,23,24,25]+ [4,6,10,13,15,17]+[1,2,9,14,]	0.97	0.87
56	7	5	[25,30,43,44,45,46]+[10,14,38,39,40,41]+ [20,29,33,34,35,36]+[11,26,13,27,28,31]+ [1,2,19,21,22,23]+[5,6,7,8,9,12]+[3,4,15,16]	0.97	0.86
70	7	5	[7,8,65,66,67,68]+[19,21,60,61,62,63]+ [31,34,55,56,57,58]+[9,11,51,50,52,53]+ [25,20,45,46,47,48]+[32,40,38,41,42,43]+ [17,18,30,33,36,37]+[2,4,24,26,27,28]+ [1,5,10,12,13,14]+[3,6,16,22]	0.98	0.86
84	7	5	[7,8,79,80,81,82]+[19,21,74,75,76,77]+ [31,34,69,70,71,72]+[2,4,64,65,66,67]+ [15,16,59,60,61,62]+[30,26,55,54,56,57]+ [1,10,35,50,51,52]+[9,13,44,45,46,47]+ [12,28,39,41,40,43]+[3,5,22,32,33,36]+ [6,20,23,25,27,29]+[11,14,17,18]	0.98	0.86
98	7	5	[7,8,93,94,95,96]+[19,21,88,89,90,91]+ [31,34,83,84,85,86]+[44,46,78,79,80,81]+ [6,11,73,74,75,76]+[20,22,68,69,70,71]+ [30,37,63,64,65,66]+[40,52,58,59,60,61]+ [9,10,53,54,55,56]+[23,26,47,48,49,50]+ [3,4,5,13,41,42,43]+[1,14,33,35,36,38]+ [15,16,17,18,24,25]+[2,12,27,28]	0.97	0.86
16	9	5	[7,5,9,10,11,12,13,14]+[1,2,3,4]	0.91	0.91
34	9	5	[18,10,27,28,29,30,31,32]+ [3,6,20,21,22,23,24,25]+ [5,4,7,9,13,14,15,16]+[1,2,8,11]	0.96	0.90
52	9	5	[18,10,45,46,47,48,49,50]+ [14,11,38,39,40,41,42,43]+ [2,20,31,32,33,34,35,36]+ [9,24,12,25,23,27,28,29]+ [1,6,8,7,13,15,16,17]+[3,4,5,19]	0.95	0.89
70	9	5	[18,10,63,64,65,66,67,68]+ [2,5,56,57,58,59,60,61]+ [26,30,49,51,50,52,53,54]+ [15,20,42,43,44,45,46,47]+ [7,34,8,36,37,38,39,40]+ [1,25,4,27,28,29,31,32]+ [3,9,13,11,14,21,22,23]+[6,12,16,17]	0.98	0.89
88	9	5	[18,10,81,82,83,84,85,86]+ [40,37,74,75,76,77,78,79]+ [21,17,67,68,69,70,71,72]+	0.97	0.89

			[8,6,45,61,62,63,64,65]+ [22,24,53,54,55,57,56,58]+ [4,5,46,47,48,49,50,51]+ [20,33,36,38,39,41,42,43]+ [19,25,28,29,30,31,32,34]+ [2,3,7,9,12,13,14,15]+ [1,11,23,26]		
18	9	7	[8,7,11,11,10,13,14,16]+[1,3,2,4,5,6]	0.88	0.72
36	9	7	[22,6,29,30,31,32,33,34]+[3,10,42,14,25,26,27]+ [11,12,15,16,17,13,19,20]+[1,2,4,5,7,8]	0.93	0.85
54		7	[18,10,47,48,49,50,51,52]+ [11,12,40,41,42,43,44,45]+ [5,13,33,35,34,36,37,38]+ [7,25,8,26,29,28,30,31]+ [1,2,3,4,15,17,19,23]+[6,9,14,16,20,21]	0.97	0.90
72	9	7	[18,10,65,66,67,68,69,70]+ [2,3,58,59,60,61,62,63]+ [26,28,51,52,53,54,55,56]+ [15,16,44,45,47,46,48,49]+ [1,7,37,38,39,40,41,42]+[4,5,12,30,31,32,33,34]+ [6,8,11,9,17,23,21,24]+[13,14,19,20,22,29]	0.98	0.90
90	9	7	[18,10,83,84,85,86,87,88]+ [33,44,76,77,78,79,80,81]+ [20,16,70,71,72,73,74]+ [42,43,62,63,64,65,66,67]+ [21,23,55,56,57,59,58,60]+ [1,2,48,49,50,51,52,53]+ [11,24,30,31,32,34,35,36]+ [3,5,15,22,25,26,27,28]+ [4,6,7,8,9,12,13,14]+[19,38,39,40,41,46]	0.98	0.87

4.2 Catalogue of efficient CWBRMDs when p_1 is odd and p_2 is even

Catalogue of efficient CWBRMDs for $v = ri+s+2$, i even, $p_1 = r$ (odd) and $p_2 = s$ (even) using $(i+1)$ sets of shifts. Here, $v< 100$, $5 \leq p_1 \leq 9$ and $4 \leq p_2 \leq 8$ is presented in Table 2.

Table 2: Efficient CWBRMDs for $v< 100$, $5 \leq p_1$ (odd) ≤ 9 and $4 \leq p_2$ (even) ≤ 8.

v	p_1	p_2	**Sets of Shifts**	**Es**	**Er**
16	5	4	[3,4,5,9]+[10,12,13,14]+[1,2,6]	0.91	0.83
26	5	4	[2,4,23,24]+[5,10,20,21]+[9,15,17,18]+ [6,7,11,12]+[1,3,8]	0.94	0.82
36	5	4	[2,4,33,34]+[7,8,30,31]+[1,23,27,28]+ [11,22,24,25]+[5,16,10,20]+[9,13,17,14]+[3,6,12]	0.96	0.81

46	5	4	[2,4,43,44]+[7,8,40,41]+[11,13,37,38]+ [16,17,34,35]+[12,30,31,32]+[3,5,27,28]+ [1,22,20,24]+[14,15,18,19]+[6,9,10]	0.97	0.81
56	5	4	[2,4,53,54]+[7,8,50,51]+[11,13,47,48]+ [16,17,44,45]+[12,30,41,42]+[14,37,38,39]+ [31,32,34,35]+[1,27,22,29]+[18,19,24,25]+ [3,6,9,15]+[5,10,20]	0.97	0.81
66	5	4	[2,4,63,64]+[7,8,60,61]+[11,13,57,58]+ [16,17,54,55]+[12,30,51,52]+[5,46,48,49]+ [43,19,44,45]+[3,10,38,40]+[6,14,36,37]+ [14,18,31,34]+[9,22,27,32]+[24,25,26,28]+ [1,20,21]	0.96	0.81
76	5	4	[2,4,73,74]+[7,8,70,71]+[11,13,67,68]+ [16,17,64,65]+[12,30,61,62]+[5,46,58,59]+ [6,54,55,56]+[22,50,51,52]+[39,45,47,48]+ [14,26,27,42]+[20,19,32,40]+[10,34,35,36]+ [15,29,31,33]+[3,9,18,21]+[1,23,24]	0.98	0.81
86	5	4	[2,4,83,84]+[7,8,80,81]+[11,13,77,78]+ [16,17,74,75]+[19,23,71,72]+[5,46,68,69]+ [6,54,65,66]+[14,58,61,62]+[,40,42,57,59]+ [30,53,55,56]+[1,18,50,51]+[3,25,47,48]+ [22,29,32,44]+[28,31,33,39]+[26,35,36,37]+ [9,12,20,21]+[10,15,27]	0.98	0.81
96	5	4	[2,4,93,94]+[7,8,90,91]+[11,13,87,88]+ [16,17,84,85]+[19,23,81,82]+[5,46,78,79]+ [6,54,75,76]+[31,38,72,73]+[10,68,69,70]+ [27,63,65,66]+[1,9,59,61]+[3,35,36,58]+ [12,18,51,55]+[15,22,50,52]+[25,26,45,47]+ [30,33,42,43]+[20,28,39,41]+[24,34,37,40]+ [14,21,29]	0.99	0.78
20	7	4	[7,8,15,16,17,18]+[2,5,6,9,11,13]+[1,3,4]	0.93	0.68
34	7	4	[7,8,29,30,31,32]+[2,24,4,25,26,27]+ [15,16,19,20,21,22]+[3,5,6,9,13,14]+[1,10,11]	0.96	0.87
48	7	4	[7,8,43,44,45,46]+[19,21,38,39,40,41]+ [5,12,33,34,35,36]+[20,22,28,29,30,31]+ [11,14,25,18,23,26]+[1,2,3,4,6,15]+[9,10,13]	0.97	0.86
62	7	4	[7,8,57,58,59,60]+[19,21,52,53,54,55]+ [1,2,47,48,49,50]+[3,25,42,43,44,45]+ [26,27,37,38,39,40]+[6,10,32,33,34,35]+ [4,5,9,24,23,29]+[11,13,15,17,18,22]+[12,14,16]	0.98	0.86
76	7	4	[7,8,71,72,73,74]+[19,21,66,67,68,69]+ [32,33,61,62,63,64]+[5,9,56,57,58,59]+ [10,29,51,52,53,54]+[30,34,46,47,48,49]+	0.98	0.86

			[2,11,41,42,43,44]+[18,23,37,36,35,39]+ [3,12,25,26,27,28]+ [1,4,6,13,15,17]+[14,16,22]		
90	7	4	[7,8,85,86,87,88]+[19,21,80,81,82,83]+ [32,33,75,76,77,78]+[42,70,48,71,72,73]+ [12,13,66,65,67,68]+[22,28,60,61,62,63]+ [35,40,55,56,57,58]+[4,6,50,51,52,53]+ [18,44,23,43,46,47]+[2,16,20,24,38,39]+ [9,29,11,30,31,34]+[1,3,5,14,15,25]+[10,17,26]	0.98	0.86
24	9	4	[17,1,3,18,19,20,21,22]+ [5,6,7,13,9,11,14,15]+[2,4,8]	0.94	0.90
42	9	4	[18,10,35,36,37,38,39,40]+ [15,20,28,30,29,31,32,33]+ [3,1,17,22,23,24,25,26]+ [2,4,6,8,9,11,12,13]+[5,7,14]	0.97	0.89
60	9	4	[18,10,53,54,55,56,57,58]+ [5,46,12,47,48,49,50,51]+ [2,39,4,40,41,42,43,44]+ [24,31,32,34,33,35,36,37]+ [8,3,11,23,25,26,27,28]+ [1,9,14,15,19,20,21]+[6,7,13]	0.96	0.86
78	9	4	[7,8,71,72,73,74,75,76]+ [30,37,64,65,66,67,68,69]+ [14,57,16,58,59,60,61,62]+ [4,9,51,50,52,53,54,55]+ [6,10,13,44,45,46,47,48]+ [20,22,34,35,38,36,40,41]+ [18,21,25,24,26,27,28,31]+ [1,2,3,5,12,17,23,29]+[11,15,19]	0.94	0.89
96	9	4	[13,15,89,90,91,92,93,94]+ [30,47,82,83,84,85,86,87]+ [14,16,75,76,77,78,79,80]+ [29,50,68,69,70,71,72,73]+ [12,20,61,63,62,64,65,66]+ [28,53,54,55,56,57,58,59]+ [18,35,44,43,45,46,49,51]+ [7,8,36,37,38,39,40,41]+ [2,6,5,22,31,27,32,33]+ [1,3,6,9,10,11,17,18]+[25,21,24]	0.97	0.89
22	7	6	[6,8,9,10,12,13]+[14,15,16,17,18,19,20]+ [1,2,3,4,5]	0.90	0.65
36	7	6	[7,8,31,32,33,34]+[1,3,26,27,28,29]+ [14,15,22,23,21,24]+[11,12,16,17,13,19]+ [2,4,5,6,9]	0.96	0.87
50	7	6	[7,8,45,46,47,48]+[19,21,40,41,42,43]+	0.93	0.87

			[5,10,36,35,37,38]+[18,22,30,31,32,33]+ [4,24,2,26,27,28]+[9,11,13,14,15,16]+ [1,2,3,6,17]		
64	7	6	[7,8,59,60,61,62]+[19,21,54,55,56,57]+ [1,2,47,50,51,52]+[10,14,44,45,46,49]+ [25,26,39,40,41,42]+[3,9,34,35,36,37]+ [18,23,28,29,31,30]+[11,13,15,16,22,24]+ [4,5,6,12,17]	0.98	0.87
78	7	6	[7,8,73,74,75,76]+[19,21,68,69,70,71]+ [30,35,63,64,65,66]+[3,9,58,59,60,61]+ [17,20,53,54,55,56]+[29,33,48,49,50,51]+ [4,5,43,44,45,46]+[2,37,34,38,41,40]+ [1,6,22,28,31,32]+[15,16,23,25,24,26]+ [10,11,12,13,14]	0.98	0.87
92	7	6	[7,8,87,88,89,90]+[19,21,82,83,84,85]+ [30,35,77,78,79,80]+[40,50,72,73,74,75]+ [11,12,67,68,69,70]+[10,38,62,63,64,65]+ [34,39,57,58,59,60]+[2,4,52,53,54,55]+ [13,31,37,47,48,49]+[29,32,41,42,43,44]+ [9,25,26,27,28,33]+[1,3,5,14,22,23]+ [6,15,16,17,18]	0.98	0.86
26	9	6	[2,1,18,20,21,22,24,23]+[3,5,4,11,14,16,15,17]+ [6,7,8,9,10]	0.94	0.91
44	9	6	[13,15,37,38,39,40,41,42]+ [12,21,30,31,32,33,34,35]+ [18,20,23,24,25,26,27,28]+ [1,2,3,5,11,14,16,17]+[4,6,7,8,9]	0.97	0.90
62	9	6	[13,15,55,56,57,58,59,60]+ [7,8,48,49,50,51,52,53]+ [1,2,40,42,43,44,45,46]+ [17,33,34,35,36,37,38,39]+ [3,5,6,26,27,28,30,29]+ [14,18,19,20,21,22,23,24]+[4,9,10,11,12]	0.98	0.90
80	9	6	[13,15,73,74,75,76,77,78]+ [33,44,66,67,68,69,70,71]+ [21,25,59,60,61,62,63,64]+ [7,8,52,53,54,55,56,57]+ [34,30,46,45,47,48,49,50]+ [12,32,37,39,36,38,41,42]+ [19,22,23,26,27,28,29,31]+ [1,2,3,5,6,9,10,20]+[4,11,14,16,17]	0.98	0.89
98	9	6	[13,15,91,92,93,94,95,96]+ [32,45,84,85,86,87,88,89]	0.99	0.89

			+[12,16,77,78,79,80,81,82]+ [55,22,70,72,71,73,74,75]+ [11,17,63,64,65,66,67,68]+ [23,54,56,57,58,59,60,61]+ [28,30,48,33,47,50,51,52]+ [3,6,29,40,41,42,43,44]+ [20,24,31,34,35,36,37,38]+ [1,2,4,9,10,14,18,19]+[5,7,8,25,26]		
28	9	8	[3,4,20,21,22,23,24,25]+[1,2,13,11,15,16,17,18]+ [5,6,7,8,9,10,12]	0.95	0.92
46	9	8	[18,10,39,40,41,42,43,44]+ [15,16,33,32,34,35,36,37]+ [14,20,25,26,27,28,29,30]+ [7,6,13,9,17,19,22,21]+[1,2,3,4,5,8,11]	0.97	0.90
64	9	8	[18,10,57,58,59,60,61,62]+ [6,7,50,51,52,53,54,55]+[1,2,38,44,45,46,47,48]+ [20,22,36,37,39,40,41,42]+ [3,4,12,16,30,31,29,33]+ [5,13,21,23,24,25,26,27]+[8,9,11,14,15,17,19]	0.98	0.90
82	9	8	[18,10,75,76,77,78,79,80]+ [11,66,68,69,70,71,72,73]+ [20,60,30,61,62,63,64,65]+ [5,13,53,54,55,56,57,58]+ [22,45,46,47,48,49,50,51]+ [9,36,38,40,37,39,42,43]+ [3,21,28,29,31,32,33,34]+ [1,2,17,19,23,24,25,26]+[4,6,7,8,12,14,15]	0.98	0.90

4. Catalogues of efficient CWBRMDs in Periods of Two Different Sizes in Which Some Ordered Pairs Appear Twice Together as Preceded values

In this Section, catalogues of efficient CWBRMDs are presented in periods of two different sizes. In these designs ordered pairs {(0, v/2), (1, (v+2)/2), …, ((v-2)/2, v-1), (v/2, 0), ((v+2)/2, 1), …, (v-1, (v-2)/2)} appear twice together while all other appear once. These designs are constructed using the following sets of shifts.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $\quad j = 1, 2, \ldots, i.$

$S_{i+1} = [q_{i1}, q_{i2}, \ldots, q_{i(s-1)}]$

Where

- $1 \leq q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{j(s-1)} \leq v-1.$

- S* contains each of 1, 2, …,v-1 exactly once except v/2 which appears twice.
- S* = [q_{j1},q_{j2}, …, $q_{j(r-1)}$, v-(q_{j1}+q_{j2}+…+$q_{j(r-1)}$), q_{i1}, q_{i2}, …, $q_{j(s-1)}$,v-(q_{i1}+q_{i2}+…+$q_{i(s-1)}$)]

5.1 Catalogue of efficient CWBRMDs When p_1 and p_2 both are odd

Catalogue of efficient CWBRMDs for $v = ri+s$, i odd, $p_1 = r$ (odd) and $p_2 = s$ (odd) using (i+1) sets of shifts. Here, v< 100, $5 \leq p_1 \leq 9$ and $3 \leq p_2 \leq 7$ in Table 3.

Table 3: Efficient CWBRMDs for v< 100, $5 \leq p_1$ (odd) ≤ 9 and $3 \leq p_2$ (odd) ≤ 7.

v	p_1	p_2	**Sets of Shifts**	**Es**	**Er**
8	5	3	[1,2,3,4]+[5,4]	0.82	0.85
18	5	3	[1,2,3,7]+[6,4,9,8]+[17,16,12,13]+[15,11]	0.92	0.81
28	5	3	[14,2,3,4]+[6,7,8,9]+[16,12,13,1]+[15,11,17,18]+ [20,27,22,19]+[25,10]	0.92	0.82
38	5	3	[24,2,3,4]+[37,7,8,9]+[27,12,13,14]+ [36,22,19,18]+[21,20,17,23]+ [25,26,6,28]+[30,35,32,1]+[31,34]	0.96	0.81
48	5	3	[34,2,3,4]+[6,7,16,9]+[42,12,13,14]+ [44,17,18,19]+[26,47,24,23]+[21,39,27,28]+ [30,38,32,33]+[35,40,36,8]+[37,41,1,43]+[45,20]	0.97	0.81
58	5	3	[44,2,3,4]+[24,7,8,9]+[57,12,13,14]+ [16,48,18,19]+[40,22,23,6]+[31,28,27,29]+ [45,30,32,33]+[46,36,37,38]+[56,41,42,43]+ [49,35,54,39]+[29,51,52,53]+[26,21]	0.98	0.81
68	5	3	[54,2,3,4]+[6,7,8,9]+[11,12,13,14]+ [16,17,15,21]+[10,31,46,24]+ [20,27,28,29]+[66,37,34,33]+ [35,36,30,64]+[22,41,42,43]+ [63,65,44,48]+[50,51,49,53]+ [55,40,57,58]+[60,61,59,45]+[23,19]	0.98	0.80
78	5	3	[64,2,3,4]+[6,7,46,9]+[24,12,13,14]+ [77,17,18,19]+[21,22,23,70]+[26,27,28,45]+ [76,32,33,58]+[74,44,39,38]+[71,41,42,43]+ [61,29,47,48]+[75,73,57,53]+[31,56,52,36]+ [60,11,66,63]+[65,72,67,68]+[8,62,69,16]+ [50,55]	0.98	0.80
88	5	3	[74,83,16,86]+[6,7,8,9]+[34,12,13,14]+ [87,17,18,19]+[76,28,23,24]+[26,27,21,36]+ [31,32,33,60]+[29,43,38,22]+[4,42,37,49]+ [45,46,47,48]+[50,51,52,57]+[55,56,53,41]+ [11,61,62,63]+[65,85,64,68]+[40,71,72,80]+	0.98	0.80

			[30,82,77,84]+[73,81,39,2]+[75,10]		
98	5	3	[84,2,3,4]+[64,7,8,9]+[44,12,13,14]+ [16,17,18,19]+[97,22,23,24]+[26,27,89,29]+ [62,32,33,34]+[36,37,38,39]+[41,42,43,21]+ [40,47,48,11]+[49,86,52,53]+[55,56,57,58]+ [60,61,20,63]+[96,95,73,59]+[70,74,72,94]+ [75,76,80,78]+[77,81,85,6]+[71,51,87,91]+ [79,88,65,93]+[66,31]	0.98	0.80
10	7	3	[2,1,3,5,4,9]+[7,8]	0.86	0.88
24	7	3	[21,2,4,5,8]+[20,9,11,19,12,13]+ [15,14,16,17,18,10]+[3,22]	0.91	0.86
38	7	3	[11,2,3,4,5,6]+[8,9,19,1,12,13]+ [15,16,17,18,37,19]+[29,36,23,24,25,26]+ [28,22,10,31,32,33]+[35,21]	0.95	0.80
52	7	3	[25,2,3,4,5,6]+[8,9,37,11,12,13]+ [15,17,16,18,19,50]+[22,23,24,1,26,27]+ [51,29,35,31,32,33]+[48,36,10,38,47,40]+ [42,43,45,44,20,39]+[30,46]	0.96	0.86
66	7	3	[39,2,3,4,5,6]+[8,9,10,11,1,13]+ [15,16,18,26,19,17]+[22,23,24,25,49,27]+ [65,36,61,37,38,12]+[43,57,44,45,46,47]+ [64,63,51,52,53,54]+[56,42,58,55,60,30]+ [35,61,37,38,12,41]+[50,20]	0.95	0.86
80	7	3	[53,2,3,4,5,6]+[8,9,10,11,15,13]+ [49,16,17,18,19,20]+[31,7924,25,26,27]+ [29,54,23,32,33,34]+[78,37,38,39,40,47]+ [42,43,44,45,46,52]+[12,50,51,40,1,30]+ [77,22,57,59,60,62]+[76,64,65,71,67,68]+ [70,55,72,73,74,75]+[66,36]	0.97	0.86
94	7	3	[67,2,3,4,5,6]+[8,9,27,11,12,13]+ [77,16,17,18,19,20]+[52,23,24,25,26,10]+ [29,88,31,32,30,34]+[36,40,35,39,49,41]+ [50, 93,46,45,47,48]+[92,44,51,22,53,54]+ [91,84,58,59,55,61]+[63,64,65,66,75,68]+ [89,71,72,73,74,15]+[70,78,90,80,81,82]+ [57,85,86,87,33,43]+[56,37]	0.98	0.86
12	9	3	[1,4,2,9,5,6,7,8]+[3,10]	0.88	0.91
30	9	3	[16,3,6,4,2,5,7,8]+[22,11,13,12,14,15,17,1]+ [29,28,20,21,10,27,24,25]+[23,19]	0.95	0.89
48	9	3	[1,2,3,4,5,6,7,8]+[11,31,9,13,15,14,16,17]+ [47,28,22,21,23,24,25,26]+ [27,20,29,30,46,32,33,34]+ [36,35,38,39,40,19,42,43]+[45,10]	0.97	0.85

66	9	3	[22,2,3,4,5,6,7,8]+[10,11,12,19,15,14,16,17]+ [13,38,21,1,23,24,25,26]+ [65,29,30,39,32,33,34,35]+ [36,37,64,49,40,41,42,43]+ [46,45,20,31,48,50,51,52]+ [54,55,56,63,58,59,60,61]+[57,47]	0.98	0.86
84	9	3	[40,2,3,4,5,6,7,8]+[52,11,12,13,14,15,16,17]+ [19,21,65,22,23,24,25,26]+ [76,29,30,31,32,33,34,35]+ [37,38,39,1,50,42,43,44]+ [45,46,47,48,49,41,81,10]+ [54,55,56,77,20,59,60,61]+ [63,64,58,66,67,69,51,79]+ [82,73,74,75,83,57,78,80]+[68,72]	0.98	0.87
12	7	5	[5,1,8,6,7,10]+[2,3,4,6]	0.88	0.90
26	7	5	[7,8,21,22,23,24]+[5,9,16,17,18,19]+ [1,2,3,4,14,13]+[6,10,11,12]	0.95	0.88
40	7	5	[7,8,35,36,37,38]+[4,12,14,31,32,33]+ [5,15,26,27,28,29]+[6,9,19,20,21,22]+ [1,3,10,11,13,17]+[2,16,18,20]	0.96	0.87
54	7	5	[7,8,49,50,51,52]+[12,28,44,45,46,47]+ [5,39,6,40,41,42]+[19,36,17,35,37,38]+ [2,15,20,29,31,32]+[13,21,22,23,26,27]+ [1,3,4,9,10,11]+[14,18,24,25]	0.97	0.87
68	7	5	[7,8,63,64,65,66]+[1,9,20,59,60,61]+ [15,45,54,55,56,57]+[10,12,48,49,50,51]+ [26,24,42,43,44,46]+[3,6,37,38,39,40]+ [14,25,31,32,33,34]+[19,18,27,28,29,30]+ [2,4,5,11,13,16]+[21,22,34,25]	0.96	0.86
82	7	5	[7,8,77,78,79,80]+[10,30,72,73,74,75]+ [32,33,67,68,69,70]+[3,5,62,63,64,65]+ [16,17,57,58,59,60]+[13,45,52,53,54,55]+ [37,46,47,48,49,50]+[15,25,39,40,41,42]+ [1,11,9,34,35,36]+[2,4,6,12,18,19]+ [14,20,22,24,26,27]+[25,23,28,29]	0.98	0.86
96	7	5	[7,8,91,92,93,94]+[10,30,86,87,88,89]+ [20,45,81,82,83,84]+[15,75,76,77,78,79]+ [2,22,70,71,72,73]+[17,32,65,66,67,68]+ [16,58,60,61,62,63]+[11,19,31,55,56,57]+ [14,26,47,48,50,51]+[1,9,13,40,42,43]+ [3,4,35,36,37,38]+[18,23,27,29,28,33]+ [5,6,12,21,41,53]+[24,25,46,48]	0.99	0.86

14	9	5	[2,3,5,8,7,9,11,12]+[1,4,6,7]	0.90	0.92
32	9	5	[12,16,25,26,27,28,29,30]+ [7,6,18,19,20,21,22,23]+[2,1,3,4,8,5,10,14]+ [9,11,13,12]	0.93	0.92
50	9	5	[18,10,43,44,45,46,47,48]+ [12,36,15,38,37,39,40,41]+ [6,20,29,30,31,32,33,34]+ [1,2,3,22,16,25,26,27]+[9,19,23,24]	0.95	0.93
68	9	5	[36,60,61,62,63,64,65,66]+ [7,9,53,54,55,56,57,58]+ [30,35,46,47,48,49,50,51]+ [21,25,39,40,41,42,43,44]+ [5,10,12,20,24,29,33,34]+ [14,17,18,22,19,23,27,26]+ [1,2,3,4,6,8,13,15]+[11,28,31,32]	0.93	0.77
86	9	5	[18,10,79,80,81,82,83,84]+ [30,47,72,73,74,75,76,77]+ [21,19,65,66,67,68,69,70]+ [1,2,58,59,60,61,62,63]+ [35,20,51,52,53,54,55,56]+ [9,42,11,44,45,46,48,49]+ [7,8,12,31,37,39,40,41]+[3,4,5,17,23,27,28,32]+ [6,13,14,15,16,24,22,25]+[26,29,36,38]	0.97	0.89
16	9	7	[7,6,8,10,11,12,13,14]+[1,2,3,4,5,8]	0.91	0.93
34	9	7	[18,10,27,28,29,30,31,32]+ [3,6,20,21,22,23,24,25]+ [4,5,7,11,13,14,15,16]+[1,2,8,9,12,17]	0.96	0.90
52	9	7	[18,10,45,46,47,48,49,50]+ [14,11,38,39,40,41,42,43]+ [2,20,31,32,33,34,35,36]+ [9,21,12,25,26,27,28,29]+ [1,6,8,7,13,15,16,17]+[3,4,5,19,23,24]	0.95	0.90
70	9	7	[18,10,63,64,65,66,67,68]+ [2,5,56,57,58,59,60,61]+ [26,30,49,51,50,52,53,54]+ [15,20,42,43,44,45,46,47]+ [6,8,35,36,37,38,39,40]+[4,1,25,27,28,29,31,32]+ [3,9,13,11,14,21,22,23]+[7,12,17,16,19,34]	0.98	0.87

88	9	7	[18,10,81,82,83,84,85,86]+ [40,37,74,75,76,77,78,79]+ [21,17,67,68,69,70,71,72]+ [8,6,45,61,62,63,64,65]+ [22,24,53,54,55,57,56,58]+ [4,5,46,47,48,49,50,51]+ [20,33,36,38,39,41,42,43]+ [19,28,11,29,31,34,35,44]+ [2,3,7,9,12,13,14,15]+[1,23,26,27,30,44]	0.96	0.90

5.2 Catalogue of efficient CWBRMDs when p_1 is odd and p_2 is even

Catalogue of efficient CWBRMDs for $v = ri+s$, i even, $p_1 = r$ (odd) and $p_2 = s$ (even) using $(i+1)$ sets of shifts. Here, $v< 100$, $5 \leq p_1 \leq 9$ and $4 \leq p_2 \leq 8$ in Table 4.

Table 4: efficient CWBRMDs for $v< 100$, $5 \leq p_1$ (odd) ≤ 9 and $3 \leq p_2$ (even) ≤ 7.

v	p_1	p_2	**Sets of Shifts**	**Es**	**Er**
14	5	4	[2,4,11,12]+[1,3,7,8]+[5,6,7]	0.90	0.83
24	5	4	[2,4,21,22]+[7,8,18,19]+[12,13,14,15]+[1,3,5,6]+ [10,11,12]	0.90	0.82
34	5	4	[2,4,31,32]+[7,8,28,29]+[10,14,25,26]+ [1,5,18,20]+[3,13,16,17]+[6,9,11,12]+[21,23,17]	0.93	0.82
44	5	4	[2,4,41,42]+[7,8,38,39]+[10,14,35,36]+ [3,5,13,33]+[19,23,29,30]+[1,6,26,27]+ [11,15,16,21]+[9,17,20,18]+[12,22,32]	0.97	0.82
54	5	4	[2,4,51,52]+[7,8,48,49]+[10,14,45,46]+ [16,17,42,43]+[19,23,39,40]+[22,29,36,37]+ [1,5,33,34]+[3,12,30,31]+[18,20,21,24]+ [6,9,11,13]+[26,27,28]	0.97	0.81
64	5	4	[5,1,61,62]+[7,8,58,59]+[10,14,55,56]+ [16,17,52,53]+[18,24,49,50]+[21,30,46,47]+ [22,38,43,44]+[2,4,39,41]+[3,19,34,35]+ [9,26,31,29]+[13,23,25,27]+[6,11,12,15]+ [36,32,28]	0.98	0.81
74	5	4	[2,4,71,72]+[7,8,68,69]+[10,14,65,66]+ [16,17,62,63]+[18,24,59,60]+[25,26,56,57]+ [20,40,53,54]+[31,38,50,51]+[1,3,47,48]+ [5,13,39,45]+[9,11,41,43]+[21,23,33,35]+ [6,19,12,15]+[28,27,29,30]+[42,37,32]	0.98	0,81

84	5	4	[2,4,81,82]+[7,8,78,79]+[10,14,75,76]+ [16,20,71,72]+[22,23,68,69]+[26,28,65,66]+ [30,33,62,63]+[36,37,58,60]+[6,56,57,59]+ [1,5,53,54]+[3,12,50,51]+[11,29,31,48]+ [15,18,44,45]+[19,32,38,39]+[9,17,21,24]+ [25,27,34,35]+[41,44,43]	0.98	0.81
94	5	4	[2,4,91,92]+[7,8,88,89]+[10,14,85,86]+ [16,17,82,83]+[20,22,79,80]+[25,26,76,77]+ [46,73 ,48 74]+[3,57,70,71]+[5,66,67,68]+ [19,62,63,64]+[37,58,59,60]+[9,18,52,53]+ [1,34,36,55]+[27,30,31,49]+[10,38,42,44]+ [21,35,40,41]+[6,14,23,28]+[15,32,47]	0.98	0.81
18	7	4	[3,2,5,6,15,7]+[8,4,11,10,12,13]+[1,9,17]	0.92	0.87
32	7	4	[21,26,27,28,29,30]+[6,17,11,25,22,23]+ [3,4,5,8,14,12]+[1,2,7,10,9,15]+[13,16,19]	0.96	0.87
46	7	4	[2,22,4,24,43,44]+[7,10,19,25,40,41]+ [11,18,13,28,37,38]+[15,16,17,31,34,35]+ [12,21,27,29,30,32]+[1,3,5,6,8,9]+[20,23,26]	0.97	0.86
60	7	4	[2,27,4,33,57,58]+[7,28,8,32,54,55]+ [11,29,13,31,51,52]+[16,25,17,35,48,49]+ [20,24,22,36,45,46]+[5,9,18,19,42,43]+ [23,21,37,38,39,40]+[1,3,6,10,12,14]+[26,30,34]	0.96	0.86
74	7	4	[7,6,32,35,71,72]+[19,11,66,67,68,69]+ [22,33,61,62,63,64]+[1,5,56,57,58,59]+ [10,21,51,52,53,54]+[26,30,46,47,48,49]+ [3,4,41,42,43,44]+[18,8,13,16,24,29]+ [2,12,14,34,27,28]+[9,15,17,20,23,25]+[36,37,38]	0.98	0.86
88	7	4	[8,24,25,37,85,86]+[21,9,80,81,82,83]+ [32,23,75,76,77,78]+[42,38,70,71,72,73]+ [5,12,65,66,67,68,]+[28,14,60,61,62,63]+ [27,40,55,56,57,58]+[3,6,45,51,52,53]+ [34,41,43,46,47,48]+[2,12,16,18,20,26]+ [7,11,29,30,31,33]+[1,4,10,15,17,19]+[49,44,39]	0.97	0.86
22	9	4	[1,2,4,3,18,12,19,20]+ [5,9,10,6,12,13,14,16]+[7,11,15]	0.87	0.90
40	9	4	[18,6,28,16,23,35,37,38]+ [15,12,36,29,30,31,32,33]+ [1,5,19,17,21,22,24,25]+ [2,3,4,7,8,9,10,11]+[13,20,27]	0.94	0.90
58	9	4	[2,7,18,46,53,54,55,56]+ [5,12,34,47,48,49,50,51]+ [4,16,20,39,40,41,42,43]+ [24,25,30,32,33,35,36,37]+ [3,6,11,19,23,26,27,28]+	0.96	0.89

			[1,9,8,10,13,15,17,20]+[14,29,44]		
76	9	4	[2,1,3,9,71,74,73,72]+[26,37,64,65,66,67,68,69]+ [20,16,57,58,59,60,61,62]+ [4,5,50,51,52,53,54,55]+ [6,14,15,40,44,45,46,47]+ [18,22,33,34,35,36,41,42]+ [7,23,24,25,30,29,27,31]+ [8,10,11,12,13,17,19,20]+[28,38,48]	0.96	0.89
94	9	4	[13,15,40,41,89,90,91,92]+ [33,30,82,83,84,85,86,87]+ [14,16,75,76,77,78,79,80]+ [29,50,69,68,70,71,72,73]+ [12,20,61,62,63,65,64,66]+ [28,53,54,55,56,57,58,59]+ [19,35,38,43,45,44,49,51]+ [7,8,24,26,34,36,37,39]+ [2,6,5,21,22,25,27,31]+ [1,3,4,9,10,11,17,18]+[46,47,48]	0.96	0.89
20	7	6	[8,10,12,16,17,18]+[9,5,11,6,13,14]+[1,3,4,7,10]	0.93	0.89
34	7	6	[7,8,29,30,31,32]+[24,2,4,25,26,27,28]+ [15,16,19,20,21,22]+[5,3,6,10,13,14]+ [1,9,11,12,17]	0.96	0.88
48	7	6	[7,8,43,44,45,46]+[19,21,38,39,40,41]+ [5,12,33,34,35,36]+[20,22,28,29,30,31]+ [11,16,18,23,24,25]+[1,2,3,4,6,15]+ [9,10,13,14,24]	0.97	0.87
62	7	6	[7,8,57,58,59,60]+[19,21,52,53,54,55]+ [1,2,47,48,49,50]+[3,25,42,43,44,45]+ [26,27,37,38,39,40]+[6,10,32,33,34,35]+ [4,5,9,24,23,29]+[11,13,15,17,18,20]+ [12,14,16,22,28]	0.97	0.87
76	7	6	[7,8,71,72,73,74]+[19,21,66,67,68,69]+ [32,33,61,62,63,64]+[5,9,56,57,58,59]+ [10,29,51,52,53,54]+[30,34,46,47,48,49]+ [2,11,41,42,43,44]+[18,24,35,36,37,38]+[3,12,25, 26,27,28]+ [1,4,6,13,15,17]+[14,16,22,23,38]	0.98	0.87
90	7	6	[7,8,85,86,87,88]+[19,21,80,81,82,83]+ [32,33,75,76,77,78]+[42,70,48,71,72,73]+ [12,13,66,65,67,68]+[22,28,60,61,62,63]+ [35,40,55,56,57,58]+[4,6,50,51,52,53]+ [17,23,43,45,46,47]+[2,16,20,24,38,39]+ [9,29,11,30,31,34]+[1,3,5,14,15,25]+ [10,18,26,37,44]	0.98	0.86

24	9	6	[8,1,12,18,19,20,21,22]+ [6,5,7,11,9,13,14,15]+[2,3,4,10,12]	0.94	0.91
42	9	6	[18,10,35,36,37,38,39,40]+ [15,20,28,29,30,31,32,33]+ [1,21,5,17,22,24,25,26]+ [2,4,6,8,9,11,12,13]+[3,7,14,16,21]	0.97	0.85
60	9	6	[18,10,53,54,55,56,57,58]+ [5,46,12,47,48,49,50,51]+ [4,16,39,30,40,41,42,43]+ [24,31,32,34,33,35,36,37]+ [8,3,11,23,25,26,27,28]+ [1,9,14,15,19,17,20,21]+[2,6,7,13,30]	0.96	0.89
78	9	6	[7,8,71,72,73,74,75,76]+ [30,37,64,65,66,67,68,69]+ [14,57,16,58,59,60,61,62]+ [4,9,51,50,52,53,54,55]+ [13,10,15,39,44,45,46,47]+ [20,22,34,35,38,36,40,41]+ [18,21,24,25,26,27,28,31]+ [1,2,3,5,12,17,23,29]+[11,6,19,33,39]	0.95	0.87
96	9	6	[13,15,89,90,91,92,93,94]+ [30,47,82,83,84,85,86,87]+ [14,16,75,76,77,78,79,80]+ [29,50,68,69,70,71,72,73]+ [12,20,61,63,62,64,65,66]+ [28,53,54,55,56,57,58,59]+ [19,35,43,44,45,46,49,51]+ [7,8,26,37,39,40,41,42]+ [2,6,5,22,31,27,32,33]+ [1,3,4,9,10,11,17,18]+[25,21,24,36,38]	0.99	0.89
26	9	8	[18,2,1,20,21,22,23,24]+ [3,5,4,12,13,16,15,17]+[6,7,8,9,10,11,13]	0.95	0.92
44	9	8	[13,15,37,38,39,40,41,42]+ [12,21,30,31,32,33,34,35]+ [18,20,23,24,25,26,27,28]+ [1,2,5,3,8,14,17,22]+[4,6,7,9,10,11,19]	0.97	0.90
62	9	8	[13,15,55,56,57,58,59]+[7,8,48,49,50,51,52,53]+ [1,2,40,42,43,44,45,46]+ [17,33,34,35,36,37,38,39]+ [3,4,6,26,27,28,29,31]+ [14,18,19,20,21,22,23,24]+[5,9,10,11,12,16,30]	0.97	0.90

80	9	8	[13,15,73,74,75,76,77,78]+ [33,44,66,67,68,69,70,71]+ [21,25,59,60,61,62,63,64]+ [7,8,52,53,54,55,56,57]+ [34,30,46,45,47,48,49,50]+ [11,32,36,37,38,40,41,42]+ [19,22,23,26,27,28,29,31]+ [1,2,3,5,6,9,10,20]+[4,12,14,16,17,18,39]	0.98	0.90
98	9	8	[13,15,91,92,93,94,95,96]+ [32,45,84,85,86,87,88,89]+ [12,16,77,78,79,80,81,82]+ [55,22,70,72,71,73,74,75,]+ [11,17,63,64,65,66,67,68]+ [23,54,56,57,58,59,60,61]+ [28,30, 47,33,48, 51,50,52]+ [3,6,26,40,41,42,43,44]+ [20,24,31,34,35,36,37,38]+ [1,2,4,9,10,14,18,19]+[5,7,8,25,27,29,46]	0.99	0.90

5. Conclusion

Catalogues of the designs are always useful for the experimenters because of their readymade solution. The catalogues presented in this article provide the all possible minimal efficient CWBRMDs for v (even) < 100, $5 \leq p_1$ (odd) ≤ 9 and $3 \leq p_2 \leq 7$ through method of cyclic shifts (Rule I). As a future research, catalogues of these designs for some of the remaining cases can also be constructed through method of cyclic shifts (Rule II).

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