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Four Component F-Squares Based Nearly D- and A-Optimal Orthogonal Block Designs for Additive Quadratic Mixture Model and Reduced Cubic Canonical Model

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ABSTRACT

Prescott (1998) considered nearly optimal orthogonal blocked designs based on latin squares for mixtures involving three and four components. Aggarwal *et al*. (2011) obtained nearly optimal orthogonally blocked designs for four mixture components based on F-squares. F-square based four component optimal orthogonal designs for an additive quadratic mixture model and the reduced cubic canonical models were obtained by Husain and Parveen (2016), and Husain and Sharma (2017), respectively. In this article, we have constructed nearly D- and A-optimal orthogonal designs in two blocks based on F-squares for mixture model involving the additive quadratic mixture model and the reduced cubic canonical model in four components.

1. Introduction

Suppose that the response depends only on the relative proportions (by weight, volume, cost, etc.) of the predictor variables x_i and not on the total amount of each independent (predictor) variable. The predictor variables are restricted to a (q -1) dimensional simplex region subject to the constraints that the x_i being component proportions add up to unity. For example, the simplex region is an equilateral triangle for q = 3 and a tetrahedron for q = 4.

F-squares are a generalisation of latin squares. Mixture designs in orthogonal blocks using F-squares were presented by Aggarwal *et al*. (2009). The best designs in terms of D-, A- and E- optimality were obtained by Aggarwal *et al*. (2009) for Scheffé's quadratic model. Aggarwal *et al*. (2008, 2013) obtained four

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component D-, A- and E-optimal orthogonal block designs in two blocks based on F-squares for Darroch and Waller's (1985) quadratic mixture model and Becker's (1968), and Draper and Pukelshiem's (1998) K-model, respectively. Aggarwal *et al*. (2011) studied nearly optimal orthogonally blocked designs for four mixture components based on F-squares. D-, A- and E- optimal designs based on F-squares for additive quadratic mixture model and reduced cubic canonical models were obtained by Husain and Parveen (2016), and Husain and Sharma (2017), respectively. Optimal designs for these models consists of binary blends with the exception of the common blend which is repeated in each block. Prescott (1998) obtained latin square based nearly optimal orthogonal block designs for Scheffé's quadratic model in three and four mixture components. The transformation suggested by Prescott (1998) modifies the designs in such a way that the orthogonality property of the considered designs is maintained but some or all of the blends contain a minimum amount of each of the required component proportions. Section 2 presents the considered mixture models, the blocking conditions and the F-square based designs. Section 3 discusses the reparameterisation of the coordinate system for F-square based designs as presented in Aggarwal *et al*. (2011). In section 4, nearly D- and A-optimal orthogonally blocked designs are obtained in four components for the additive quadratic mixture model and the reduced cubic canonical model.

2. Mixture Models, Blocking Conditions and F-Square Based Designs

Husain and Parveen (2016) presented the additive quadratic mixture model (2.1) and obtained the conditions (2.2) for the orthogonal blocking of the mixture blends.

$$\eta_{q,1}(x) = \beta_1 x_1 + \cdots + \beta_q x_q + \beta_{12} x_1 (x_1 - x_2) + \cdots + \beta_{q-1,q} x_{q-1}(x_{q-1} - x_q) + \gamma z_u + \varepsilon_u$$

$$i,\, j = 1, 2, \ldots, q \quad , \quad i \neq j \tag{2.1}$$

$$\sum_{u=1}^{n_w} x_{u_i} = k_i, \qquad i = 1, 2, \ldots, q$$

$$\sum_{u=1}^{n_w} x_{u_i}(x_{u_i} - x_{u_j}) = k_{ij}, \qquad i \neq j = 1, 2, \ldots, q, i \neq j \tag{2.2}$$

Scheffé's full cubic canonical model is as given in (2.3).

$$\eta_{q,2}(x) = \sum_{i=1}^{q} \beta_i x_i + \sum_{1 \le i < j \le q} \beta_{ij} x_i x_j + \sum_{1 \le i < j \le q} \beta_{ij} x_i x_j (x_i - x_j) + \sum_{1 \le i < j \le k} \beta_{ijk} x_i x_j x_k \tag{2.3}$$

Husain and Sharma (2017) presented the reduced cubic canonical model (2.4) with block effect and obtained the orthogonality conditions given in (2.5).

$$\eta_{q,3}(x) = \sum_{i=1}^{q} \beta_i x_i + \sum_{1 \le i < j \le q} \beta_{ij} x_i x_j \left| \; x_i - x_j \; \right| + \gamma Z_u + \varepsilon_u \,, \; i = 1,2,\ldots,q, \; u = 1,2,\ldots,n$$

$$\sum_{u=1}^{n_w} x_{u_i} = k_i \,, \qquad\qquad i = 1,2,\ldots,q \qquad\qquad (2.4)$$

$$\sum_{u=1}^{n_w} x_{u_i} x_{u_j} \left| \; x_{u_i} - x_{u_j} \; \right| = k_{ij} \,, \qquad\qquad i \neq j = 1,2,\ldots,q \qquad\qquad (2.5)$$

For four component mixtures fitted to the additive quadratic mixture model (2.1) and the reduced cubic canonical model (2.4), nine distinct runs are needed in order to ascertain the estimability of the unknown parameters. Aggarwal *et al*. (2009) suggested the class of designs given in Table 1 based on F-squares for Scheffé's quadratic model in four components. These designs have 18 runs in two blocks. Each block contains nine runs representing the specific four component mixtures.

Table 1. Design 1, Design 2 and Design 3 in two blocks

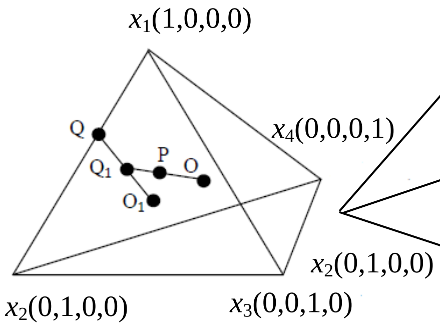
Design 1		Design 2		Design 3	
B_1	B_2	B_1	B_2	B_1	B_2
$\begin{bmatrix} a & b & c & a \\ b & c & a & a \\ c & a & a & b \\ a & a & b & c \\ a & c & a & b \\ b & a & a & c \\ c & a & b & a \\ a & b & c & a \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} a & a & c & b \\ b & a & a & c \\ c & b & a & a \\ a & c & b & a \\ a & c & b & a \\ b & a & c & a \\ c & a & a & b \\ a & b & a & c \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} a & b & c & a \\ b & c & a & a \\ c & a & a & b \\ a & a & b & c \\ a & a & b & c \\ b & a & c & a \\ c & b & a & a \\ a & c & a & b \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} a & a & c & b \\ b & a & a & c \\ c & b & a & a \\ a & c & b & a \\ a & b & a & c \\ b & c & a & a \\ c & a & b & a \\ a & a & c & b \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} a & c & a & b \\ b & a & a & c \\ c & a & b & a \\ a & b & c & a \\ a & a & b & c \\ b & a & c & a \\ c & b & a & a \\ a & c & a & b \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} a & c & b & a \\ b & a & c & a \\ c & a & a & b \\ a & b & a & c \\ a & b & a & c \\ b & c & a & a \\ c & a & b & a \\ a & a & c & b \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

This class of design consists of 13 distinct quaternary blends. In this paper, we have used this class of design to obtain the nearly D- and A-optimal orthogonal block designs for the additive quadratic mixture model (2.1) and the reduced cubic canonical model (2.4) in four components. Optimal designs for additive quadratic mixture model $\eta_{q,1}$ and the reduced cubic canonical model $\eta_{q,3}$ consist of binary blends with the exception of the quaternary blend (1/4, 1/4, 1/4, 1/4) which is added to both the blocks to remove singularity of the design matrix.

Practically, there are situations where all the ingredients need to be physically a part of the mixture. For example, milk, water, sugar and tea leaves are all required in making the tea beverage. In such situations, rather than using optimal designs, a way out is to lose some efficiency and obtain nearly optimal designs consisting of pure mixtures as suggested by Prescott (1998).

3. Reparameterisation of the Coordinate System

Prescott (1998) suggested reparameterisation of the coordinates in order to simplify the optimization criteria and studied the properties of the alternative designs in three and four components obtained by shrinking the optimal designs towards the centroid. For $q = 4$, the reparameterisation for a point P (a, b, c, d) with $a \geq b \geq c \geq d$ takes the form of two shrinkages. Firstly, a point Q is shrunk by an amount s_1 on the edge x_3, x_4 towards the centroid of the $x_4 = 0$ face to the point Q_1. Secondly, point Q_1 is shrunk by an amount s_2 towards the centroid of the simplex.



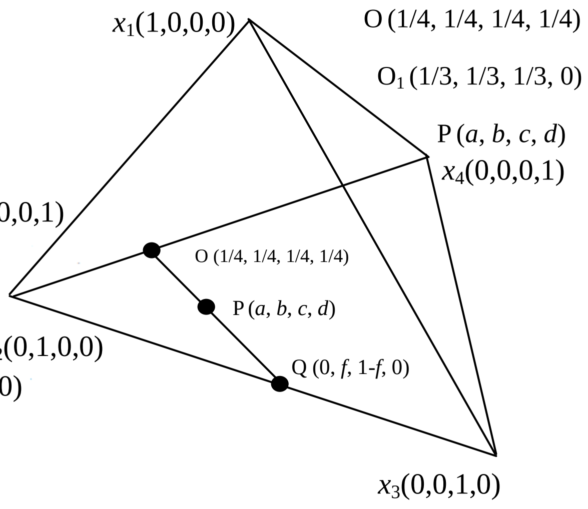


Figure1. Reparameterisation of P from (a, b, c, d) with $a \geq b \geq c \geq d$ to (f, s_1, s_2).

Figure 2. Reparameterisation of P from (a, b, c, d) with $b \geq c \geq d = a$ to (f, s).

These shrinkages are illustrated in Figure 1 and the resulting reparameterisation is given by Prescott (1998) as follows:

$$a = (1 - s_2)(1 - s_1)f + (1 - s_2)s_1/3 + s_2/4$$

$$b = (1 - s_2)(1 - s_1)(1 - f) + (1 - s_2)s_1/3 + s_2/4$$

$$c = (1 - s_2)s_1/3 + s_2/4$$

$$d = s_2/4 \qquad (3.1)$$

The coordinates of P at (a, b, c, d) are now in terms of (f, s_1, s_2). In general, this reparameterisation does not represent a simple shrinkage of the simplex towards its centroid. For F-square based designs, P is such that $x_1 = x_4$ (i.e. $a = d$), then $s_1 = 0$ and the above reparameterisation follows simple shrinkage of the simplex towards its centroid. Aggarwal *et al*. (2011) obtained the following co-ordinates of P for F-square based designs.

$$b = (1 - s)f + s/4$$
$$c = (1 - s)(1 - f) + s/4$$

$$a = d = s/4 \qquad (3.2)$$

The point P is now in terms of f and s only as shown in Figure 2.

4. Nearly Optimal Orthogonally Blocked Designs For Models $\eta_{q,1}$ and $\eta_{q,3}$

In this section, we present nearly optimal orthogonal designs based on F- squares for Husain and Parveen's (2016) additive quadratic mixture model and Husain and Sharma's (2017) reduced cubic canonical model in four components, respectively. In order to obtain nearly D- and A-optimal designs, we first present the general expressions for $|\mathbf{X}'\mathbf{X}|$ for model $\eta_{q,1}$ and $\eta_{q,3}$ in (4.1) and (4.2) as obtained by Husain and Parveen (2016) and Husain and Sharma (2017), respectively. For both the considered models, the general expressions of T are very lengthy and hence not discussed here.

$$|\mathbf{X}'\mathbf{X}| = 20736\,(a - c)^{10}\,(a - b)^{10}\,(b - c)^{6}\,(4a^2 - 4ab - 4ac + 3b^2 - 2bc + 3c^2)^2 \qquad (4.1)$$

$$|\mathbf{X}'\mathbf{X}| = 2304(a-b)^{10}(b-c)^{6}(ab+ac+bc-c^2)^{4}(2a^2-ab-3ac-bc+c^2)^{6}$$
$$(2a^2b-2ab^2+2a^2c+b^2c-2ac^2-bc^2)^2 \qquad (4.2)$$

4.1 Additive Quadratic Mixture Model $\eta_{q,1}$

Husain and Parveen (2016) obtained optimal designs involving binary blends with the exception of the overall centroid point. The general form of the $|\mathbf{X}'\mathbf{X}|$ given in (4.1) is maximized at the point $b = c$, $1 - c$ for which $|\mathbf{X}'\mathbf{X}|$ transforms to the following:

$$|\mathbf{X}'\mathbf{X}|_0 = 20736\, b^{10}\,(1-2b)^6\,(-1+b)^{10}\,(3+8(-1+b)\,b)^2 \tag{4.1.1}$$

At $b = 0.240117, 0.759885$, $|\mathbf{X}'\mathbf{X}|_0$ takes a maximum value of 0.0000396722. On applying the transformation given in (3.2) to the coordinates (a, b, c, d) for the points in Design 1, Design 2 and Design 3, the general form of the determinant in terms of f and s is as given in (4.1.2).

$$|\mathbf{X}'\mathbf{X}| = 20736\, f^{10}\,(1-2f)^6\,(-1+f)^{10}\,(3+8(-1+f)\,f)^2\,(-1+s)^{30} \tag{4.1.2}$$

Since the general design may be regarded as a shrinkage of the design with $a = 0$, $b = f$, $c = 1- f$ by a factor s, the form of equation (4.1.2) is a simple reduction in scale towards the centroid. For any fixed value of s, we observe that $|\mathbf{X}'\mathbf{X}|$ is maximized for $f = 0.240117, 0.759885$ and as $s \to 1$, $|\mathbf{X}'\mathbf{X}|$ becomes a strictly decreasing function of s which means that $|\mathbf{X}'\mathbf{X}|$ has ridges of maxima along the lines joining the centroid to the D-optimal design point on the edges of the simplex. Thus, a nearly optimal orthogonal block design may be obtained by shrinking the optimal design towards its centroid. In terms of D-criterion, the efficiency of the nearly optimal design D = $|\mathbf{X}'\mathbf{X}|^{1/p}$, where p is the number of parameters is given by D-efficiency = $|\mathbf{X}'\mathbf{X}|^{1/p} / |\mathbf{X}'\mathbf{X}|^{1/p}_{a=0} \times 100$ %

Thus, the efficiency of the nearly optimal designs relative to the D-optimal designs for model $\eta_{q,1}$ is $(1 - s)^{30/11} \times 100$. These efficiencies are shown in Table2. We observe that the D-efficiency of Husain and Parveen's model (2016) for F-square based designs Design 1, Design 2 and Design 3 matches with Prescott (1998) for Scheffé's quadratic model applied to Draper *et al's* (1993) latin square based designs and also with Husain and Hafeez's (2018) four component latin square based design for Husain and Parveen's (2016) model.

Table 3 presents nearly D-optimal block Design 1 with $f = 0.240117$ and $s = 0.05$ for model $\eta_{q,1}$. Hence, with a little loss in D-efficiency, we are able to obtain orthogonal block designs for the additive quadratic mixture model having pure mixture components. For Design 1 and Design 3, on applying the transformation given in (3.2), the general form of T in terms of f and s is as given in (4.1.3).

Table 2. Efficiency of the nearly D- optimal Design 1, Design 2 and Design 3 against the shrinkage parameter s for model $\eta_{q,1}$.

s	D-efficiency
0.05	86.94
0.10	75.02
0.15	64.1
0.20	54.4
0.25	45.6

$$T = \left(2+\frac{1359+(-1+f)f(11811+4(-1+f)f(9115+10572(-1+f)f))}{72(1-2f)^2(-1+f)^2 f^2(3+8(-1+f)f)^2(-1+s)^4} + \frac{7}{72(-1+f)^2 f^2(-1+s)^2}+\frac{11+14(-1+f)f}{48(-1+f)^2 f^2(-1+s)^2} \right) \quad (4.1.3)$$

Table 3. Nearly D-optimal block Design 1with f = 0.240117 and s = 0.05 for model $\eta_{q,1}$.

B$_1$				**B**$_2$			
0.0125	0.240611	0.734389	0.0125	0.0125	0.0125	0.734389	0.240117
0.240611	0.734389	0.0125	0.0125	0.240117	0.0125	0.0125	0.734389
0.734389	0.0125	0.0125	0.240611	0.734389	0.240117	0.0125	0.0125
0.0125	0.0125	0.240611	0.734389	0.0125	0.734389	0.240117	0.0125
0.0125	0.734389	0.0125	0.240611	0.0125	0.734389	0.240117	0.0125
0.240611	0.0125	0.0125	0.734389	0.240117	0.0125	0.734389	0.0125
0.734389	0.0125	0.240611	0.0125	0.734389	0.0125	0.0125	0.240117
0.0125	0.240611	0.734389	0.0125	0.0125	0.240117	0.0125	0.734389
0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25

For different values of s, we get T as a function of f. For s = 0, T is minimum at f = 0.265523, 0.734477. The general form of T in terms of f and s for Design 2 is given in (4.1.4).

$$T = \left(2+\frac{891+2(-1+f)f(3879+4(-1+f)f(3001+3492(-1+f)f))}{48(1-2f)^2(-1+f)^2 f^2(3+8(-1+f)f)^2(-1+s)^4} + \frac{1}{9(-1+f)^2 f^2(-1+s)^3}+\frac{11+14(-1+f)f}{48(-1+f)^2 f^2(-1+s)^2} \right) \quad (4.1.4)$$

For different values of s, we get T as a function of f. For s = 0, T is minimum at f = 0.265814, 0.734186. Chan *et al*. (2001) gave the formula for finding the efficiency of A- optimal designs as

A-efficiency = T_0 / Min (T) × 100

Again, T_0 is the minimum T which is obtained by substituting optimal f in the original T. Table 4 presents the efficiency of nearly A-optimal Design 1 and Design 3 against the shrinkage parameter s for model $\eta_{q,1}$ while the Table 5 presents the same for Design 2.

Table 4. Efficiency of the nearly A-optimal Design 1 and Design 3 against the shrinkage parameter *s* for model $\eta_{q,1}$.

s	Optimum *f*	Min(T)	T_0	A-efficiency
0	0.265523,0.734477	107.009	107.009	100
0.05	0.265228,0.734772	130.538	107.009	81.97
0.15	0.264694,0.735306	201.486	107.009	53.10
0.20	0.264455,0.735545	255.627	107.009	41.86

Table 5. Efficiency of the nearly A-optimal Design 2 against the shrinkage parameter *s* for model $\eta_{q,1}$.

s	Optimum *f*	Min(T)	T_0	A-efficiency
0	0.265814,0.734186	104.435	104.435	100
0.05	0.265525,0.734475	127.401	104.435	81.97
0.15	0.265006,0.734994	196.664	104.435	53.10
0.20	0.264775,0.735225	249.527	104.435	41.85

Table 6. Nearly A-optimal block Design 1with *f* = 0.265228 and *s* = 0.05 for model $\eta_{q,1}$.

B$_1$				**B**$_2$			
0.0125	0.264751	0.710248	0.0125	0.0125	0.0125	0.710248	0.264751
0.264751	0.710248	0.0125	0.0125	0.264751	0.0125	0.0125	0.710248
0.710248	0.0125	0.0125	0.264751	0.710248	0.264751	0.0125	0.0125
0.0125	0.0125	0.264751	0.710248	0.0125	0.710248	0.264751	0.0125
0.0125	0.710248	0.0125	0.264751	0.0125	0.710248	0.264751	0.0125
0.264751	0.0125	0.0125	0.710248	0.264751	0.0125	0.710248	0.0125
0.710248	0.0125	0.264751	0.0125	0.710248	0.0125	0.0125	0.264751
0.0125	0.264751	0.710248	0.0125	0.0125	0.264751	0.0125	0.710248
0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25

Hence with little loss in A-efficiency, we are able to obtain true mixtures which contain some proportion of all the ingredients for the additive quadratic mixture model $\eta_{q,1}$.

Table 7. Nearly A-optimal block Design 2 with f = 0.265525 and s = 0.05 for model $\eta_{q,1}$.

$\mathbf{B}_1$				$\mathbf{B}_2$			
0.0125	0.264748	0.710251	0.0125	0.0125	0.0125	0.710251	0.26474
0.264748	0.710251	0.0125	0.0125	0.264748	0.0125	0.0125	0.71025
0.710251	0.0125	0.0125	0.26474	0.710251	0.264748	0.0125	0.0125
0.0125	0.0125	0.264748	0.71025	0.0125	0.710251	0.264748	0.0125
0.0125	0.0125	0.264748	0.71025	0.0125	0.264748	0.0125	0.71025
0.264748	0.0125	0.710251	0.0125	0.264748	0.710251	0.0125	0.0125
0.710251	0.264748	0.0125	0.0125	0.710251	0.0125	0.264748	0.0125
0.0125	0.710251	0.0125	0.26474	0.0125	0.0125	0.710251	0.26474
0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25

4.2 Reduced Cubic Canonical Model $\eta_{q,3}$

The general form of the $|\mathbf{X}'\mathbf{X}|$ given in (4.2) as obtained by Husain and Sharma (2017) is maximized at the point $b = c$, $1 - c$ for which $|\mathbf{X}'\mathbf{X}|$ transforms to the following.

$$|\mathbf{X}'\mathbf{X}|_{a=0} = 2304b^{10}(-1+2b)^6((1-b)^2-(1-b)b)^6(-(1-b)^2+(1-b)b)^4(-(1-b)^2b+(1-b)b^2)^2 \quad 4.2.1$$

$|\mathbf{X}'\mathbf{X}|_{a=0}$ takes a maximum value of 8.11923×10^{-11} at b = 0.172673, 0.827505. On applying the transformation given in (3.2) to the coordinates (a, b, c, d) for the points in Design 1, Design 2 and Design 3, the general form of the determinant in terms of f and s is very lengthy and is available with the authors. Since the general design may be regarded as a shrinkage of the design with a = 0, $b = f$, c = $1 - f$ by a factor s, the general form of the determinant in terms of f and s is a simple reduction in scale towards its centroid. For any fixed value of s, we observe that $|\mathbf{X}'\mathbf{X}|$ is maximized for f = 0.172673, 0.827505 and as $s \to 1$, $|\mathbf{X}'\mathbf{X}|$ becomes a strictly decreasing function of s which means that $|\mathbf{X}'\mathbf{X}|$ has ridges of maxima along the lines joining the centroid to the D-optimal design point on the edges of the simplex. Thus, a nearly optimal orthogonal block design may be obtained by shrinking the optimal design towards its centroid. The efficiency of the nearly D-optimal designs for model $\eta_{q,3}$ are as given in Table 8.

We observe from Table 8 that both $|\mathbf{X}'\mathbf{X}|$ and $|\mathbf{X}'\mathbf{X}|_{a=0}$ are maximum at f = 0.172678, 0.834385, 0.854724, 0.861949, 0.848632 and 0.832611, for s = 0, 0.05, 0.10, 0.15, 0.20 and 0.25, respectively. Table 9 presents nearly D-optimal block Design 1with f = 0.834385 and s = 0.05 for model $\eta_{q,3}$.

Table 8. Efficiency of the nearly D-optimal Design 1, Design 2 and Design 3 against the shrinkage parameter s for model $\eta_{q,3}$.

s	Optimum f	$\lvert \mathbf{X}'\mathbf{X} \rvert$	$\lvert \mathbf{X}'\mathbf{X} \rvert_{a=0}$	D-efficiency
0	0.172678, 0.82752	8.11923×10^{-11}	8.11923×10^{-11}	100
0.05	0.834385	5.54283×10^{-11}	7.99942×10^{-11}	96.39
0.10	0.854724	2.79719×10^{-11}	6.41638×10^{-11}	92.03
0.15	0.861949	1.00104×10^{-11}	5.53259×10^{-11}	84.28
0.20	0.848632	2.29512×10^{-11}	7.05788×10^{-11}	32.51
0.25	0.832611	1.89406×10^{-11}	8.05212×10^{-11}	23.52

Table 9. Nearly D-optimal block Design 1with f = 0.834385 and s = 0.05 for model $\eta_{q,3}$.

$\mathbf{B}_1$				$\mathbf{B}_2$			
0.0125	0.805165	0.197593	0.0125	0.0125	0.0125	0.197593	0.805165
0.805165	0.197593	0.0125	0.0125	0.805165	0.0125	0.0125	0.197593
0.197593	0.0125	0.0125	0.805165	0.197593	0.805165	0.0125	0.0125
0.0125	0.0125	0.805165	0.197593	0.0125	0.197593	0.805165	0.0125
0.0125	0.197593	0.0125	0.805165	0.0125	0.197593	0.805165	0.0125
0.805165	0.0125	0.0125	0.197593	0.805165	0.0125	0.197593	0.0125
0.197593	0.0125	0.805165	0.0125	0.197593	0.0125	0.0125	0.805165
0.0125	0.805165	0.197593	0.0125	0.0125	0.805165	0.0125	0.197593
0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25

Table 10. Efficiency of the nearly A- optimal Design 1, Design 2 and Design 3 against the shrinkage parameter s for model $\eta_{q,3}$.

s	Optimum f	Min(T)	T_0	A-efficiency
0	0.192039, 0.807961	775.656	775.656	100
0.05	0.179036, 0.804064	901.52	776.184	86.09
0.10	0.170179, 0.797244	1169.97	779.627	66.69

0.15	0.163184, 0.790973	1674.7	785.62	46.93
0.20	0.156158, 0.787194	2620.41	790.553	30.16
0.25	0.1480, 0.705947	4483.97	792.401	17.67

Table 11. Nearly A-optimal block Design 1with f = 0.179036 and s = 0.05 for model $\eta_{q,3}$.

$\mathbf{B}_1$	$\mathbf{B}_2$
$\begin{bmatrix} 0.0125 & 0.182584 & 0.792415 & 0.0125 \\ 0.182584 & 0.792415 & 0.0125 & 0.0125 \\ 0.792415 & 0.0125 & 0.0125 & 0.182584 \\ 0.0125 & 0.0125 & 0.182584 & 0.792415 \\ 0.0125 & 0.792415 & 0.0125 & 0.182584 \\ 0.182584 & 0.0125 & 0.0125 & 0.792415 \\ 0.792415 & 0.0125 & 0.182584 & 0.0125 \\ 0.0125 & 0.182584 & 0.792415 & 0.0125 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$	$\begin{bmatrix} 0.0125 & 0.0125 & 0.792415 & 0.182584 \\ 0.182584 & 0.0125 & 0.0125 & 0.792415 \\ 0.792415 & 0.182584 & 0.0125 & 0.0125 \\ 0.0125 & 0.792415 & 0.182584 & 0.0125 \\ 0.0125 & 0.792415 & 0.182584 & 0.0125 \\ 0.182584 & 0.0125 & 0.792415 & 0.0125 \\ 0.792415 & 0.0125 & 0.0125 & 0.182584 \\ 0.0125 & 0.182584 & 0.0125 & 0.792415 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$

On applying the transformation (3.2), the general forms of T in terms of f and s for Design 1, Design 2 and Design 3, are very lengthy. For different values of s, we get T as a function of f. For s = 0, T is minimum at f = 0.192039, 0.807961. Again T_0 is the minimum T which is obtained by substituting optimal f in the original T. We observe that T is minimum at f = 0.192039, 0.179036, 0.170179, 0.163184, 0.156158 and 0.1480 for s = 0, 0.05, 0.10, . . ., 0.25, respectively. T_0 is minimum at f = 0.192039, 0.807961, 0.804064, 0.797244, 0.790973, 0.787194 and 0.705947 for s = 0, 0.05, 0.10, . . ., 0.25, respectively. From tables 10 and 11, we see that when f = 0.179036 and s = 0.05, then with a little loss in A-efficiency, we get a true mixture which contains some proportion of all the ingredients.

5. Conclusions

Optimal designs consist of binary blends with the exception of a common blend (usually the centroid) which is repeated in each block. In this paper, we have applied Prescott's (1998) approach to the case when two of the four mixture components are at the same level of composition and have obtained nearly optimal orthogonal designs for Husain and Parveen's (2016) additive quadratic mixture model, and Husain and Sharma's (2017) reduced cubic canonical model. For the additive quadratic mixture model, the D- and A- efficiencies of the nearly optimal designs at $s = 0.05$ are 86.94% and 81.97%, respectively. For the reduced cubic canonical model, the D- and A- efficiencies of the nearly optimal designs are 96.39% and 86.09%, respectively at $s = 0.05$. Note that the orthogonality property of the designs is not disturbed by following Prescott's (1998) idea and with some loss in efficiencies, we are able to obtain designs consisting of quaternary blends.

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