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A Catalogue of Minimal Circular Balanced Repeated Measurements Designs in Periods of Two Different Sizes

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ABSTRACT

Repeated measurements designs are widely used in medicine, pharmacology, animal sciences and psychology. In these fields, there are several situations where these designs should be used in different period sizes. Therefore, in this article, minimal circular balanced repeated measurements designs are constructed in periods of two different sizes where period sizes are smaller than v (number of treatments). Catalogues of these designs are presented for v up to 100, where $5 \leq p_1$ (odd) ≤ 9, $2 \leq p_2 \leq 8$, and $p_1 > p_2$.

1. Introduction

A repeated measurement design (RMD) is one in which multiple, or repeated measurements are made on each experimental unit. The experimental unit could be a person or an animal. Areas where RMDs are widely used include medicine, pharmacology, animal sciences and psychology. A key feature of RMD is that the effect which a treatment has during its period of application (its direct effect) may persist into the following periods (residual effect). The choice of RMD must be made in a way that the treatments can be efficiently compared after allowing for the residual effects. It can be done by using balanced repeated measurements designs (BRMDs). RMD is balanced with respect to the first-order residual

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effects if each treatment is immediately preceded λ' times by each other treatment (excluding itself). For given v and p, a BRMD is minimal if $\lambda' = 1$.

Magda (1980) introduced the idea of a circular balanced repeated measurements design (CBRMD) when proper balance for different effects is considered. Afsarinejad (1990) presented some construction methods for BRMDs. Afsarinejad (1994) gave an easy method of constructing minimal balanced and strongly balanced RMDs with unequal period sizes. Using method of cyclic shifts developed by Iqbal (1991), Iqbal and Jones (1994) constructed (i) efficient RMDs with equal and unequal period sizes, (ii) Strongly BRMDs for two unequal period sizes. Sharma *et al.* (2003) introduced a general strategy of construction of BRMDs for odd number of treatments and their analysis. Iqbal and Tahir (2009) constructed CSBRMDs for some classes using method of cyclic shifts. Iqbal *et al*. (2010) constructed some first- and second-order CBRMD and circular strongly balanced RMDs (CSBRMDs). In this paper, CBRMD are constructed in periods of two different sizes where $p_2 < p_1 < v$. Bailey *et al*. (2017) constructed universally optimal weakly BRMDs for $p = v$. Khan *et al*. (2018) developed some infinite series to obtain circular weakly balanced repeated measurements designs (CWBRMDs) for (i) $v = 2mi+2$, $p = 2m$, m (> 1) integer, (ii) $v = 2m(i+1)-1$, $p = 2m$, m (> 1) integer, (iii) $v = 2mi+3$, $p = 2m$, m (> 1) integer, and $v = 2mi+4$, $p = 2m$, m (> 1) integer. They also constructed CWBRMDs for (i) $v = ri+2$, i even and $p = r > 3$ (odd), (ii) $v = ri+r-1$, i odd and $p = r > 3$ (odd), (iii) $v = ri+3$, i odd and $p = r > 3$ (odd), and $v = ri+3$, i even and $p = r > 3$ (odd). Bashir *et al*. (2018) presented procedures to construct CBRMDs for some cases. Rajab *et al*. (2018) developed some infinite series to generate the CBRMDs in equal period sizes. Rasheed *et al*. (2018) developed some infinite series to generate minimal CSBRMDs in periods of three different sizes $p_1 < p_2 < p_3$.

The rest of the paper is organized as follows: To construct CBRMDs, method of cyclic shifts is explained in Section 2. In Section 3, some new CBRMDs are constructed for v up to 100, where $5 \le p_1 \text{ (odd)} \le 9$, $2 \le p_2 \le 8$.

2. Method of Cyclic Shifts

How a design is obtained from a given set(s) of shifts using Rule I, is explained here. For it, a design is obtained through set of shifts [3,4,9] for $v = 12$ and $p = 4$. Take v subjects for set of shifts [3,4,9]. Assign 0, 1,…, v-1 to each subject in first period. To get the elements of second period for each subject, add 3 (mod v) of [3,4,9] to each element of first period for all subjects. Then add 4 (mod v) to each element of second period for all subjects of third period. Similarly add 9.

Periods ↓	**Subjects**											
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	0	1	2
3	7	8	9	10	11	0	1	2	3	4	5	6
4	4	5	6	7	8	9	10	11	0	1	2	3

How a CBRMD is obtained for p_1 and p_2 from given sets of shifts using Rule I, is explained here.

Let $S_1 = [q_{11}, q_{12}, \ldots, q_{1(a-1)}]$ and $S_2 = [q_{21}, q_{22}, \ldots, q_{2(b-1)}]$ be sets of shifts, where $1 \leq q_{ij} \leq v\text{-}1$, $p_1 = a$ and $p_2 = b$ If each element 1, 2, …,v -1 appears an equal number of times, say λ' in a new set of shifts S* then it will be CBRMD in periods of sizes p_1 and p_2. Here S* = $[q_{11}, q_{12}, \ldots, q_{1(a-1)}, q_{21}, q_{22}, \ldots, q_{2(b-1)}, v\text{-}(q_{11}+q_{12}+\ldots+q_{1(a-1)}) \pmod v, v\text{-}(q_{21}+q_{22}+\ldots+q_{2(b-1)} \pmod v)]$.

Example 2.1. Sets of shifts [3,4,7,5]+[1,2] provide following CBRMD for $v = 9$, $p_1 = 5$ and $p_2 = 3$.

Subjects								
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0_1	1_2	2_3	3_4	4_5	5_6	6_7	7_8	8_0
3_0	4_1	5_2	6_3	7_4	8_5	0_6	1_7	2_8
7_3	8_4	0_5	1_6	2_7	3_8	4_0	5_1	6_2
5_7	6_8	7_0	8_1	0_2	1_3	2_4	3_5	4_6
1_5	2_6	3_7	4_8	5_0	6_1	7_2	8_3	0_4

10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
0_3	1_4	2_5	3_6	4_7	5_8	6_0	7_1	8_2
1_0	2_1	3_2	4_3	5_4	6_5	7_6	8_7	0_8
3_1	4_2	5_3	6_4	7_5	8_6	0_7	1_8	2_0

How a design is obtained from a given set(s) of shifts using Rule II, is explained here. For it, a design is obtained through sets of shifts $S_1 = [2,4,5,6]$ and $S_2 = [1,3]$t for $v = 9$, $p_1 = 5$ and $p_2 = 4$.

Take v-1 subjects for set of shifts [2,4,5,6]. Assign 0, 1,…, v-2 to each subject in first period. To get elements of second period for each subject, add 2 mod (v-1) of [2,4,5,6] to each element of first period for all subjects. Then add 4, 5 and 6 in similar manner.

Periods	**Subjects**							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0	1	2	3	4	5	6	7
2	2	3	4	5	6	7	0	1
3	6	7	0	1	2	3	4	5
4	3	4	5	6	7	0	1	2
5	1	2	3	4	5	6	7	0

Take v-1 subjects for one set of shifts [1,3]t. Assign 0, 1, …, v-2 to each subject in first period. For elements of 2nd period for each subject, add 1 mod (v-1) of [1,3]t to each element of first period for all subjects. Then add 3 mod (v-1) to each element of 2nd period for all subjects. Insert v-1 = 8 for all subjects of 4th period.

Periods	**Subjects**							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0	1	2	3	4	5	6	7
2	1	2	3	4	5	6	7	0
3	4	5	6	7	0	1	2	3
4	8	8	8	8	8	8	8	8

How a CBRMD is obtained for p_1 and p_2 from given set(s) of shifts using Rule II, is explained here.

Let $S_1 = [q_{11}, q_{12}, \ldots, q_{1(a\text{-}1)}]$ and $S_2 = [q_{21}, q_{22}, \ldots, q_{2(b\text{-}2)}]$t be sets of shifts, where $1 \leq q_{ij} \leq v\text{-}2$, $p_1 = a$ and $p_2 = b$. If each element 1, 2, …,v-2 appears an equal number of times, say λ' in a new set of shifts S* then it will be CBRMD in periods of sizes p_1 and p_2. If at least one set of shifts is of type $[q_{21}, q_{22}, \ldots, q_{2(b\text{-}2)}]$t then $S^* = [q_{11}, q_{12}, \ldots, q_{1(a\text{-}1)}, q_{21}, q_{22}, \ldots, q_{2(b\text{-}2)}, v\text{-}1\text{-}(q_{11}+q_{12}+ \ldots+q_{1(a\text{-}1)}) \bmod (v\text{-}1)]$.

Example 2.2. Sets of shifts $S_1 = [2,4,5,6]$, $S_2 = [1,3]$t provide following CBRMD for $v = 9$ in $p_1 = 5$ and $p_2 = 4$.

Periods	**Subjects**															
										0	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6	7	0	8	8	8	8	8	8	8	8
2	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
3	2	3	4	5	6	7	0	1	1	2	3	4	5	6	7	0

4	6	7	0	1	2	3	4	5	4	5	6	7	0	1	2	3
5	3	4	5	6	7	0	1	2								

3. Construction of CBRMDS in P_1 and P_2 By Rule I

In this section minimal CBRMDs are constructed in periods of sizes p_1 & p_2 where $p_2 < p_1 < v$, using method of cyclic shifts (Rule I).

Series 3.1. Minimal CBRMDs can be constructed with $p_1 = r$ & $p_2 = 2$, $r > 1$(odd) for $v = ri+2$; i even with $\lambda' = 1$ through the following i sets of shifts for p_1 and one set for p_2.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i$. $S_{i+1} = [v/2](1/2)$,

Where:

- $1 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)} \le v$-1 but $\ne v/2$,
- $q_{j1}+q_{j2}+ \ldots + q_{j(r-1)} \ne 0 \pmod v$,
- S* contains each of 1, 2, …, v-1 exactly once.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v/2, v - (q_{j1}+q_{j2}+ \ldots + q_{j(r-1)}) \bmod v]$

Example 3.1. Minimal CBRMD is constructed for $v = 12$ in $p_1 = 5$ and $p_2 = 2$ through the following sets of shifts.

$S_1 = [4,5,7,9]$, $S_2 = [1,2,3,8]$, $S_3 = [6](1/2)$

Table A1, A2, and A3 in Appendix A consist of designs constructed from the Series 3.1, respectively in (i) $p_1 = 5$, $p_2 = 2$, (ii) $p_1 = 7$, $p_2 = 2$, (iii) $p_1 = 9$, $p_2 = 2$ for $v \le 100$.

Series 3.2. Minimal CBRMDs can be constructed with $p_1 = r$, $p_2 = 2$, $r > 3$ (odd) for $v = ri+3$; i even with $\lambda' = 1$ through the following i sets of shifts for p_1 and two sets for p_2.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i$. $S_{i+1} = [(v-1)/2]$,

Where:

- $1 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)} \le v$-1 but $\ne (v-1)/2$ and $(v+1)/2$,
- $q_{j1}+q_{j2}+ \ldots + q_{j(r-1)} \ne 0 \pmod v$,
- S* contains each of 1, 2, …, v-1 exactly once.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, (v-1)/2, (v+1)/2, v-(q_{j1}+q_{j2}+ \ldots + q_{j(r-1)}) \bmod v]$

Example 3.2. Minimal CBRMD is constructed for $v = 13$ in $p_1 = 5$ and $p_2 = 2$ through the following sets of shifts.

$$\mathbf{S_1} = [2,7,10,11],\ \mathbf{S_2} = [3,5,4,8],\quad \mathbf{S_3} = [12]$$

Table A4, A5, and A6 in Appendix A consist of designs constructed from the Series 3.2, respectively in (i) $p_1 = 5$, $p_2 = 2$, (ii) $p_1 = 7$, $p_2 = 2$, (iii) $p_1 = 9$, $p_2 = 2$ for $v \leq 100$.

Series 3.3. Minimal CBRMDs can be constructed with $p_1 = r$, $p_2 = s$, $r > 3$ (odd) for $v = ri+s$; $s > 3$, i & s both are either odd or even, with $\lambda' = 1$ through the following i sets of shifts for p_1 and one set for p_2.

$S_j = [q_{j1,}\, q_{j2},\, \ldots, q_{j(r-1)}]$; $\quad j = 1, 2, \ldots, i.$

$S_{i+1} = [q_{i1,}\, q_{i2},\, \ldots,\, q_{i(s-1)}]$

Where:

- $1 \leq q_{j1,}\, q_{j2},\, \ldots,\, q_{j(r-1)},\, q_{i1,}\, q_{i2},\, \ldots,\, q_{i(s-1)} \leq v\text{-}1$,
- $q_{j1}+q_{j2}+\ldots+q_{j(r-1)} \neq 0 \pmod v$, $q_{i1}+q_{i2}+\ldots+q_{i(s-1)} \neq 0 \pmod v$,
- S* contains each of 1, 2, …,v-1 exactly once.
- S* = $[q_{j1,}\, q_{j2},\, \ldots,\, q_{j(r-1)},\, v\text{-}(q_{j1}+q_{j2}+\ldots+q_{j(r-1)}) \bmod v,\, q_{i1,}\, q_{i2},\, \ldots,\, q_{i(s-1)},$
  - $v\text{-}(q_{i1}+q_{i2}+\ldots+q_{i(s-1)}) \bmod v]$

Example 3.3. Minimal CBRMD is constructed for $v = 15$ in $p_1 = 5$ and $p_2 = 4$ through the following sets of shifts.

$$\mathbf{S_1} = [4,5,6,7],\quad \mathbf{S_2} = [10,11,12,13],\quad \mathbf{S_3} = [1,2,3]$$

Table A7, Table A8, …, Table A18 in Appendix A consist of designs constructed from the Series 3.4 respectively in (i) $p_1 = 5$, $p_2 = 3$, (ii) $p_1 = 7$, $p_2 = 3$, (iii) $p_1 = 9$, $p_2 = 3$, (iv) $p_1 = 5$, $p_2 = 4$, (v) $p_1 = 7$, $p_2 = 4$, (vi) $p_1 = 9$, $p_2 = 4$, (vii) $p_1 = 7$, $p_2 = 5$, (viii) $p_1 = 9$, $p_2 = 5$, (ix) $p_1 = 7$, $p_2 = 6$, (x) $p_1 = 9$, $p_2 = 6$, (xi) $p_1 = 9$, $p_2 = 7$, and (xii) $p_1 = 9$, $p_2 = 8$ for $v \leq 100$.

4. Construction of CBRMDS in P_1 and P_2 by Rule II

Series 4.1. Minimal CBRMDs can be constructed with $p_1 = r$, $p_2 = s$, $r > 3$ (odd) and $s > 3$ (integer) for $v = ri+s$; i odd and s even (s odd and i even) with $\lambda' = 1$ through the following i sets of shifts for p_1 and one set for p_2.

$S_j = [q_{j1,}\, q_{j2},\, \ldots, q_{j(r-1)}]$; $\quad j = 1, 2, \ldots, i.$

$S_{i+1} = [q_{i1,}\, q_{i2},\, \ldots,\, q_{i(s-2)}]$t

Where:

- $1 \leq q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{i(s-2)} \leq v\text{-}2$,
- $q_{j1}+q_{j2}+\ldots+q_{j(r-1)} \neq 0 \pmod{v\text{-}1}$, $q_{i1+}q_{i2}+\ldots+q_{i(s-2)} \neq 0 \bmod (v\text{-}1)$,
- S* contains each of 1, 2, …,v-2 exactly once.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v\text{-}1\text{-}(q_{j1}+q_{j2}+\ldots+q_{j(r-1)}), q_{i1}, q_{i2}, \ldots, q_{i(s-2)}, v\text{-}1\text{-}(q_{i1+}q_{i2}+\ldots+q_{i(s-2)})]$

Example 4.1: Minimal CBRMD is constructed for $v = 19$ in $p_1 = 7$ and $p_2 = 5$ through the following sets of shifts.

$\mathbf{S_1}$ = [7,8,13,14,15,16], $\mathbf{S_2}$ = [3,4,5,9,10,11], $\mathbf{S_3}$ = [1,2,6]t

Table B1, Table B2, …, Table B9 in Appendix B consist of designs constructed from the above series respectively for (i) $p_1 = 5$, $p_2 = 4$, (ii) $p_1 = 7$, $p_2 = 4$, (iii) $p_1 = 9$, $p_2 = 4$, (iv) $p_1 = 7$, $p_2 = 5$, (v) $p_1 = 9$, $p_2 = 5$, (vi) $p_1 = 7$, $p_2 = 6$, (vii) $p_1 = 9$, $p_2 = 6$, (viii) $p_1 = 9$, $p_2 = 7$, and (ix) $p_1 = 9$, $p_2 = 8$ for $v \leq 100$.

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Appendix A

Table A1:CBRMDs for $v = 5i + 2$, i even, $p_1 = 5$ and $p_2 = 2$, $\lambda' = 1$

	Set of Shifts
2	[4,5,7,9]+[1,2,3,8]+[6](1/2)
2	[2,4,19,20]+[5,10,16,17]+[1,6,8,14]+[3,7,9,12]+[11](1/2)
2	[2,4,29,30]+[5,10,26,27]+[11,13,23,24]+[14,19,20,21]+[6,8,15,17]+ [1,3,7,9]+[16](1/2)
2	[2,4,39,40]+[5,10,36,37]+[11,13,33,34]+[14,19,30,31]+[20,22,27,28]+ [3,6,24,25]+[12,15,16,18]+[1,7,8,9]+[21](1/2)
2	[2,4,49,50]+[5,10,46,47]+[11,13,43,44]+[14,19,40,41]+[20,22,37,38]+ [24,27,34,35]+[1,7,31,32]+[8,9,28,29]+[17,18,21,23]+[3,6,12,15]+ [26](1/2)

2	[2,4,59,60]+[5,10,56,57]+[11,13,53,54]+[14,19,50,51]+[20,22,47,48]+ [24,27,44,45]+ [28,32,41,42]+[33,36,38,39]+[6,12,34,35]+[18,21,26,29]+ [3,8,9,17]+[1,7,15,16]+[31](1/2)

Table A2:CBRMDs for $v = 7i + 2$, i even, $p_1 = 7$ and $p_2 = 2$, $\lambda' = 1$

	Set of Shifts
6	[5,10,11,12,13,14]+[1,2,3,4,6,7]+[8](1/2)
0	[5,10,25,26,27,28]+[4,6,20,21,22,23]+[3,8,9,16,17,18]+ [1,2,7,11,12,13]+[15](1/2)
4	[5,10,39,40,41,42]+[7,33,34,35,36,37]+[12,14,28,29,30,31]+ [1,6,23,24,25,26]+[2,4,9,13,19,20]+[3,8,11,15,16,17]+[22](1/2)
8	[5,10,53,54,55,56]+[7,33,48,49,50,51]+[30,35,43,44,45,46]+ [15,17,38,39,40,41]+[1,3,31,32,34,36]+[21,23,24,25,26,27]+ [9,12,16,18,19,20]+[2,4,6,8,11,13]+[29](1/2)

Table A3:CBRMDs for $v = 9i + 2$, i even, $p_1 = 9$ and $p_2 = 2$, $\lambda' = 1$

	Set of Shifts
0	[3,5,13,14,15,16,17,18]+[1,2,4,6,7,8,9,11]+[10](1/2)
8	[1,2,5,7,33,34,35,36]+[8,9,11,12,28,29,30,31]+[3,4,6,14,23,24,25,26]+ [10,13,15,16,17,18,20,21]+[19](1/2)
6	[1,2,5,7,51,52,53,54]+[8,9,11,12,46,47,48,49]+[4,6,26,29,41,42,43,44]+ [19,20,21,30,36,37,38,39]+[3,14,15,27,31,32,33,34]+[10,13,16,17,18,22, 23,24]+[28](1/2)

Table A4:CBRMDs for $v = 5i + 3$, i even, $p_1 = 5$ and $p_2 = 2$, $\lambda' = 1$

	Set of Shifts
3	[2,4,10,11]+[1,3,5,8]+[6]
3	[2,4,20,21]+[5,10,17,18]+[3,6,7,14]+[1,8,9,13]+[11]
3	[2,4,30,31]+[5,10,27,28]+[11,13,24,25]+[14,19,21,22]+[6,7,15,18]+ [1,3,8,9]+[16]
3	[2,4,40,41]+[5,10,37,38]+[11,13,34,35]+[16,17,31,32]+[19,23,28,29]+ [1,7,25,26]+[9,15,18,20]+[3,6,8,12]+[21]
3	[2,4,50,51]+[5,10,47,48]+[11,13,44,45]+[16,17,41,42]+[20,22,38,39]+ [23,28,35,36]+ [29,31,32,33]+[12,15,24,25]+[1,3,9,19]+[6,7,8,14]+[26]
3	[2,4,60,61]+[5,10,57,58]+[11,13,54,55]+[16,17,51,52]+[20,22,48,49]+ [25,26,45,46]+[27,33,42,43]+[34,35,39,40]+[7,8,36,37]+[15,24,28,29]+ [1,6,12,21]+[3,9,14,18]+[31]

Table A5:CBRMDs for $v = 7i + 3$, i even, $p_1 = 7$ and $p_2 = 2$, $\lambda' = 1$

	Set of Shifts
7	[5,10,12,13,14,15]+[1,2,3,4,6,7]+[8]
1	[5,10,26,27,28,29]+[3,6,21,22,23,24]+[1,4,14,17,18,19]+[2,7,8,9,11,12]+ [15]
5	[5,10,40,41,42,43]+[19,21,35,36,37,38]+[9,11,30,31,32,33]+ [4,8,13,26,27,28]+ [15,16,17,18,20,24]+[1,2,3,6,7,12]+[22]
9	[5,10,54,55,56,57]+[19,21,49,50,51,52]+[31,34,44,45,46,47]+ [15,16,39,40,41,42]+[25,32,33,35,36,37]+[6,8,12,13,24,27]+ [1,2,3,4,7,20]+[9,11,14,17,18,23]+[29]

Table A6:CBRMDs for $v = 9i + 3$, i even, $p_1 = 9$ and $p_2 = 2$, $\lambda' = 1$

	Set of Shifts
1	[1,2,5,7,16,17,18,19]+[3,4,6,8,9,12,13,14]+[10]
9	[1,2,5,7,34,35,36,37]+[8,9,11,12,29,30,31,32]+[3,4,6,13,24,25,26,27]+ [10,14,15,16,17,18,21,22]+[19]
7	[1,2,5,7,52,53,54,55]+[8,9,11,12,47,48,49,50]+[4,6,27,28,42,43,44,45]+ [3,16,35,36,37,38,39,40]+[10,13,22,25,29,30,32,33]+ [14,15,17,18,19,20,21,23]+[26]

Table A7: CBRMDs for $v = 5i + 4$, i odd, $p_1 = 5$ and $p_2 = 3$, $\lambda' = 1$

	Set of Shifts
	[3,4,5,7]+[1,2]
9	[3,4,15,17]+[8,10,12,13]+[5,6,7,9]+[1,2]
9	[3,4,25,27]+[8,10,22,23]+[13,14,19,20]+[5,6,12,17]+[7,9,11,15]+[1,2]
9	[3,4,35,37]+[8,10,32,33]+[13,14,29,30]+[16,20,26,27]+[6,11,12,24]+ [5,7,21,22]+ [9,15,17,18]+[1,2]
9	[3,4,45,47]+[8,10,42,43]+[13,14,39,40]+[17,19,36,37]+[22,23,33,34]+ [26,28,30,31]+ [5,12,25,27]+[6,7,9,11]+[15,18,20,21]+[1,2]
9	[3,4,55,57]+[8,10,52,53]+[13,14,49,50]+[17,19,46,47]+[20,25,43,44]+ [26,28,40,41]+[31,32,37,38]+[6,7,34,35]+[15,16,24,30]+[5,9,11,12]+ [18,21,23,27]+[1,2]
9	[3,4,65,67]+[8,10,62,63]+[13,14,59,60]+[17,19,56,57]+[20,25,53,54]+ [26,28,50,51]+[31,32,47,48]+[35,37,44,45]+[5,23,27,41]+[9,12,38,39]+ [11,24,33,34]+[15,21,29,30]+ [6,7,16,18]+[1,2]

Table A8: CBRMDs for $v = 7i + 4$, i odd, $p_1 = 7$ and $p_2 = 3$, $\lambda' = 1$

	Set of Shifts
1	[3,4,5,6,7,9]+[1,2]
5	[8,10,19,20,21,23]+[9,11,14,15,16,17]+[3,4,5,6,7,12]+[1,2]
9	[8,10,33,34,35,37]+[22,23,28,29,30,31]+[16,17,21,24,25,26]+ [3,5,6,7,18,19]+ [4,9,11,12,13,14]+[1,2]
3	[8,10,47,48,49,51]+[22,23,42,43,44,45]+[34,36,37,38,39,40]+ [25,26,30,31,32,33]+[14,16,21,24,27,28]+[4,13,15,17,18,19]+ [3,5,6,7,9,11]+[1,2]
7	[8,10,61,62,63,65]+[22,23,56,57,58,59]+[34,36,51,52,53,54]+ [13,15,46,47,48,49]+[26,27,41,42,43,44]+[5,7,35,37,38,39]+ [21,25,29,30,31,32]+[11,14,18,19,20,24]+ [3,4,6,9,12,16]+[1,2]

Table A9: CBRMDs for $v = 9i + 4$, i odd, $p_1 = 9$ and $p_2 = 3$, $\lambda' = 1$

	Set of Shifts
3	[3,4,5,6,7,8,9,11]+[1,2]
1	[3,4,5,6,25,26,27,29]+[9,10,11,15,20,21,22,23]+ [7,8,12,13,14,16,17,18]+[1,2]
9	[3,4,5,6,43,44,45,47]+[9,10,11,15,38,39,40,41]+ [7,20,21,22,33,34,35,36]+[19,23,26,27,28,29,30,31]+ [8,12,13,14,16,17,18,24]+[1,2]
7	[3,4,5,6,61,62,63,65]+[9,10,11,15,56,57,58,59]+ [7,20,21,22,51,52,53,54]+[19,23,26,27,46,47,48,49]+ [17,24,39,40,41,42,43,44]+[8,16,31,33,34,35,36,37]+[12,13,14,18,25,28,29,30]+[1,2]

Table A10:CBRMDs for $v = 5i + 5$, i even, $p_1 = 5$ and $p_2 = 4$, $\lambda' = 1$

	Set of Shifts
5	[4,5,6,7]+[10,11,12,13]+[1,2,3]
5	[15,16,22,23]+[4,12,18,20]+[5,6,8,14]+[7,9,10,11]+[1,2,3]
5	[11,13,14,33]+[5,7,30,31]+[9,15,26,27]+[16,17,23,24]+[4,6,19,20]+ [8,10,12,18]+[1,2,3]
5	[15,16,17,43]+[5,7,40,41]+[10,14,36,37]+[13,20,33,34]+[19,23,30,31]+ [25,26,27,28]+ [11,12,21,22]+[4,6,8,9]+[1,2,3]
5	[18,19,21,53]+[5,7,50,51]+[9,15,46,47]+[16,17,43,44]+[20,22,40,41]+ [25,26,37,38]+[29,31,34,35]+[4,11,30,32]+[8,23,24,27]+ [6,10,12,13]+[1,2,3]

5	[21,23,24,63]+[5,7,60,61]+[9,15,56,57]+[16,17,53,54]+[20,22,50,51]+ [25,26,47,48]+[29,31,44,45]+[34,35,41,42]+[6,8,37,39]+[11,12,33,36]+ [13,27,28,30]+[4,10,14,18]+[1,2,3]

Table A11: CBRMDs for $v = 7i + 5$, i odd, $p_1 = 7$ and $p_2 = 4$, $\lambda' = 1$

	Set of Shifts
9	[7,8,14,15,16,17]+[4,5,6,9,10,11]+[1,2,3]
3	[7,8,28,29,30,31]+[4,9,21,23,24,25]+[5,6,10,17,19,20]+ [11,12,13,14,15,16]+[1,2,3]
7	[7,8,42,43,44,45]+[22,23,36,37,38,39]+[11,12,31,32,33,34]+ [4,10,13,27,28,29]+ [9,16,20,21,24,25]+[5,6,14,15,17,18]+[1,2,3]
1	[7,8,56,57,58,59]+[22,23,50,51,52,53]+[34,36,45,46,47,48]+ [16,18,40,41,42,43]+[30,32,33,35,37,38]+[17,25,26,27,28,29]+ [9,14,15,19,20,21]+[4,5,6,10,11,12]+[1,2,3]

Table A12:CBRMDs for $v = 9i + 5$, i even, $p_1 = 9$ and $p_2 = 4$, $\lambda' = 1$

	Set of Shifts
3	[5,8,9,16,18,19,20,21]+[4,6,7,10,11,12,13,14]+[1,2,3]
1	[5,8,9,34,36,37,38,39]+[17,19,27,28,29,30,31,32]+ [6,7,10,21,22,23,24,25]+ [4,11,12,13,14,15,16,18]+[1,2,3]
9	[5,8,9,52,54,55,56,57]+[20,21,22,46,47,48,49,50]+ [29,31,39,40,41,42,43,44]+[23,27,32,33,34,35,36,37]+ [4,12,13,15,24,25,26,28]+[6,7,10,11,14,16,17,18]+[1,2,3]

Table A13:CBRMDs for $v = 7i + 6$, i odd, $p_1 = 7$ and $p_2 = 5$, $\lambda' = 1$

	Set of Shifts
7	[7,8,22,23,24,25]+[5,9,16,18,19,20]+[6,10,11,12,13,14]+[1,2,3,4]
1	[7,8,36,37,38,39]+[20,21,30,32,33,34]+[14,15,25,26,27,28]+ [5,12,18,19,22,23]+[6,9,10,11,13,16]+[1,2,3,4]
5	[7,8,50,51,52,53]+[20,21,44,46,47,48]+[34,36,39,40,41,42]+ [22,23,32,33,35,37]+[9,11,27,28,29,30]+[5,6,10,14,24,25]+ [12,13,15,16,17,18]+[1,2,3,4]
9	[7,8,64,65,66,67]+[20,21,58,60,61,62]+[34,36,53,54,55,56]+ [12,14,48,49,50,51]+[25,26,43,44,45,46]+[13,15,17,39,40,41]+ [10,22,32,33,35,37]+[6,9,16,18,28,30]+ [5,11,19,23,24,27]+[1,2,3,4]

Table A14:CBRMDs for $v = 9i + 6$, i odd, $p_1 = 9$ and $p_2 = 5$, $\lambda' = 1$

	Set of Shifts
5	[6,7,8,9,10,11,12,13]+[1,2,3,4]
3	[6,7,8,27,28,29,30,31]+[5,9,13,20,21,22,24,25]+ [10,11,12,14,15,16,17,18]+[1,2,3,4]
1	[6,7,8,45,46,47,48,49]+[11,12,37,38,39,40,42,43]+ [10,14,30,31,32,33,34,35]+[9,13,23,24,25,26,27,28]+ [5,15,16,17,18,19,20,21]+[1,2,3,4]
9	[6,7,8,63,64,65,66,67]+[11,12,37,56,57,58,60,61]+ [5,10,15,50,51,52,53,54]+[9,14,43,44,45,46,47,48]+ [17,18,20,21,38,39,40,41]+[24,25,26,31,32,33,34,35]+ [13,16,19,22,23,27,28,29]+[1,2,3,4]

Table A15:CBRMDs for $v = 7i + 7$, i even, $p_1 = 7$ and $p_2 = 6$, $\lambda' = 1$

	Set of Shifts
1	[7,8,16,17,18,19]+[9,10,11,12,13,14]+[1,2,3,4,5]
5	[7,8,30,31,32,33]+[19,21,25,26,27,28]+[10,11,12,15,17,18]+ [6,9,13,14,16,23]+[1,2,3,4,5]
9	[7,8,44,45,46,47]+[19,21,39,40,41,42]+[6,11,33,35,36,37]+ [22,24,28,29,30,31]+[12,14,20,23,25,26]+[9,10,13,15,16,17]+[1,2,3,4,5]
3	[7,8,58,59,60,61]+[19,21,53,54,55,56]+[32,34,47,49,50,51]+ [15,17,42,43,44,45]+[22,35,37,38,39,40]+[14,16,29,30,31,33]+ [9,11,12,13,26,27]+[6,10,18,20,23,24]+[1,2,3,4,5]

Table A16: CBRMDs for $v = 9i + 7$, i even, $p_1 = 9$ and $p_2 = 6$, $\lambda' = 1$

	Set of Shifts
5	[6,7,8,19,20,21,22,23]+[9,11,12,13,14,15,16,17]+[1,2,3,4,5]
3	[6,7,8,37,38,39,40,41]+[18,19,20,31,32,33,34,35]+ [9,22,23,24,25,26,27,29]+[10,11,12,13,14,15,16,17]+[1,2,3,4,5]
1	[6,7,8,55,56,57,58,59]+[18,19,20,49,50,51,52,53]+ [31,32,34,42,43,44,45,47]+[17,22,35,36,37,38,39,40]+ [9,11,12,15,16,28,29,30]+[10,13,14,21,23,24,25,26]

Table A17: CBRMDs for $v = 9i + 8$, i odd, $p_1 = 9$ and $p_2 = 7$, $\lambda' = 1$

	Set of Shifts
7	[7,8,9,10,11,12,14,15]+[1,2,3,4,5,6]
5	[10,12,13,15,30,31,32,33]+[11,17,23,24,25,26,27,28]+ [7,8,9,16,18,19,20,21]+[1,2,3,4,5,6]

3	[11,18,19,20,48,49,50,51]+[13,17,21,42,43,44,45,46]+ [25,27,35,36,37,38,39,40]+[8,9,10,28,29,30,31,33]+ [7,12,14,15,16,22,23,24]+[1,2,3,4,5,6]

Table A18:CBRMDs for $v = 9i + 9$, i even, $p_1 = 9$ and $p_2 = 8$, $\lambda' = 1$

	Set of Shifts
7	[9,10,11,17,21,22,23,24]+[8,12,13,14,15,16,18,19]+[1,2,3,4,5,6,7]
5	[13,14,15,18,40,41,42,43]+[10,20,21,34,35,36,37,38]+ [11,12,19,28,29,30,31,32]+[8,9,16,22,23,24,25,26]+[1,2,3,4,5,6,7]
3	[18,19,20,21,58,59,60,61]+[14,15,22,52,53,54,55,56]+ [17,25,45,46,47,48,49,50]+[12,16,38,39,40,41,42,43]+ [9,10,30,31,32,33,34,36]+[8,11,13,23,24,26,27,28]+[1,2,3,4,5,6,7]

Appendix B

Table B1: CBRMDs for $v = 5i + 4$, i odd, $p_1 = 5$ and $p_2 = 4$, $\lambda' = 1$

	Set of Shifts
	[2,4,5,6]+[1,3]t
9	[2,4,15,16]+[6,9,12,13]+[3,5,7,10]+[1,8]t
9	[2,4,25,26]+[7,8,22,23]+[10,14,19,20]+[3,5,15,16]+[6,9,11,12]+[1,13]t
9	[2,4,35,36]+[7,8,32,33]+[10,14,29,30]+[16,17,26,27]+[20,22,23,24]+ [9,12,15,19]+[3,5,6,11]+[1,18]t
9	[2,4,45,46]+[7,8,42,43]+[11,13,39,40]+[16,17,36,37]+[20,22,33,34]+ [25,26,30,31]+[3,9,27,28]+[14,18,19,21]+[5,6,10,12]+[1,23]t
9	[2,4,55,56]+[7,8,52,53]+[11,13,49,50]+[16,17,46,47]+[20,22,43,44]+ [25,26,40,41]+[29,31,37,38]+[5,6,34,35]+[12,18,23,30]+[14,19,24,27]+ [3,9,10,15]+[1,28]t
9	[2,4,65,66]+[6,9,62,63]+[11,13,59,60]+[16,17,56,57]+[20,22,53,54]+ [25,26,50,51]+[29,31,47,48]+[34,35,44,45]+[38,40,41,42]+[10,14,36,37]+ [23,24,27,30]+[5,8,15,19]+ [3,7,12,18]+[1,33]t

Table B2:CBRMDs for $v = 7i + 4$, i odd, $p_1 = 7$ and $p_2 = 4$, $\lambda' = 1$

	Set of Shifts
1	[2,3,5,6,7,8]+[1,4]t
5	[7,8,19,20,21,22]+[6,10,14,15,16,17]+[2,3,4,5,9,12]+[1,11]t
9	[7,8,33,34,35,36]+[19,21,28,29,30,31]+[10,17,23,24,25,26]+ [3,4,5,6,16,20]+ [2,9,11,12,13,14]+[1,18]t

3	[7,8,47,48,49,50]+[19,21,42,43,44,45]+[32,33,37,38,39,40]+ [20,22,30,31,34,35]+[10,12,24,26,27,28]+[2,13,15,16,17,18]+ [3,4,5,6,9,11]+[1,25]t
7	[7,8,61,62,63,64]+[19,21,56,57,58,59]+[31,34,51,52,53,54]+ [11,13,46,47,48,49]+[24,25,41,42,43,44]+[3,5,36,37,38,39]+ [20,23,28,29,30,33]+[10,12,17,18,22,26]+[2,4,6,9,14,15]+[1,32]t

Table B3:CBRMDs for $v = 9i + 4$, i odd, $p_1 = 9$ and $p_2 = 4$, $\lambda' = 1$

	Set of Shifts
13	[2,3,4,6,7,8,9,10]+[1,5]t
1	[6,7,8,24,25,26,27,28]+[5,10,12,18,19,20,21,22]+ [2,3,4,9,11,13,15,16]+[1,14]t
9	[6,7,8,42,43,44,45,46]+[18,19,20,36,37,38,39,40]+ [14,15,16,30,31,32,33,34]+[10,11,12,24,25,26,27,28]+ [2,3,4,5,9,13,17,21]+[1,23]t
7	[6,7,8,60,61,62,63,64]+[18,19,20,54,55,56,57,58]+ [29,31,33,48,49,50,51,52]+[17,22,24,42,43,44,45,46]+ [30,34,35,36,37,38,39,40]+[2,3,4,5,12,25,26,27]+ [9,10,11,13,14,15,16,21]+[1,32]t

Table B4:CBRMDs for $v = 7i + 5$, i even, $p_1 = 7$ and $p_2 = 5$, $\lambda' = 1$

	Set of Shifts
9	[7,8,13,14,15,16]+[3,4,5,9,10,11]+ [1,2,6]t
3	[7,8,27,28,29,30]+[19,21,22,23,24,25]+[4,6,15,16,17,18]+ [3,5,9,10,11,12]+[1,2,13]t
7	[7,8,41,42,43,44]+[19,21,36,37,38,39]+[9,10,31,32,33,34]+ [22,23,25,27,28,29]+[3,4,5,12,18,24]+[6,11,13,14,15,16]+[1,2,20]t
1	[7,8,55,56,57,58]+[19,21,50,51,52,53]+[32,33,45,46,47,48]+ [14,16,40,41,42,43]+[26,29,35,36,37,38]+[15,17,25,28,30,31]+ [9,10,12,20,22,23]+[3,4,5,6,11,13]+[1,2,27]t

Table B5:CBRMDs for $v = 9i + 5$, i even, $p_1 = 7$ and $p_2 = 5$, $\lambda' = 1$

	Set of Shifts
3	[5,6,10,16,17,18,19,20]+[3,4,7,9,11,12,13,14]+[1,2,8]t
1	[6,7,8,34,35,36,37,38]+[18,19,20,28,29,30,31,32]+ [5,9,10,11,23,24,25,26]+[3,4,12,13,14,15,16,21]+[1,2,17]t
9	[6,7,8,52,53,54,55,56]+[18,19,20,46,47,48,49,50]+ [30,31,32,40,41,42,43,44]+[22,24,25,34,35,36,37,38]+ [4,5,11,14,23,27,28,29]+[3,9,10,12,13,15,16,17]+[1,2,26]t

Table B6:CBRMDs for $v = 7i + 6$, $i > 1$(odd) $p_1 = 7$ and $p_2 = 6$, $\lambda' = 1$

	Set of Shifts
7	[6,9,21,22,23,24]+[4,10,16,17,18,19]+[5,8,11,12,13,14]+[1,2,3,7]t
1	[7,8,35,36,37,38]+[19,21,30,31,32,33]+[12,13,25,26,27,28]+ [4,9,18,20,22,23]+[5,6,10,11,15,16]+[1,2,3,14]t
5	[7,8,49,50,51,52]+[18,22,44,45,46,47]+[32,33,39,40,41,42]+ [17,19,34,35,36,37]+[6,11,27,28,29,30]+[4,5,9,15,24,25]+ [10,12,13,14,16,20]+[1,2,3,21]t
9	[7,8,63,64,65,66]+[19,21,58,59,60,61]+[32,33,53,54,55,56]+ [44,46,48,49,50,51]+[25,29,41,42,43,45]+[5,22,26,36,37,38]+ [23,24,27,30,31,34]+[12,14,16,17,18,20]+[4,6,9,10,11,13]+[1,2,3,28]t

Table B7:CBRMDs for $v = 9i + 6$, $i > 1$(odd) $p_1 = 9$ and $p_2 = 6$, $\lambda' = 1$

	Set of Shifts
3	[6,7,8,26,27,28,29,30]+[5,9,11,20,21,22,23,24]+ [4,12,13,14,15,16,17,18]+[1,2,3,10]t
1	[6,7,8,44,45,46,47,48]+[17,18,22,38,39,40,41,42]+ [13,14,16,32,33,34,35,36]+[4,5,20,26,27,28,29,30]+ [9,10,11,12,15,21,23,24]+[1,2,3,19]t
9	[6,7,8,62,93,64,65,66]+[18,19,20,56,57,58,59,60]+ [30,31,32,50,51,52,53,54]+[17,21,23,44,45,46,47,48]+ [29,33,35,38,39,40,41,42]+[4,5,10,25,26,27,34,36]+ [9,11,12,13,14,15,16,22]+[1,2,3,28]t

Table B8:CBRMDs for $v = 9i + 7$, $i > 2$(even) $p_1 = 9$ and $p_2 = 7$, $\lambda' = 1$

	Set of Shifts
3	[6,7,8,36,37,38,39,40]+[18,19,20,30,31,32,33,34]+ [14,16,21,24,25,26,27,28]+[5,9,10,12,13,15,17,22]+[1,2,3,4,11]t
1	[6,7,8,54,55,56,57,58]+[17,19,21,48,49,50,51,52]+ [30,31,32,42,43,44,45,46]+[22,23,24,36,37,38,39,40]+ [11,18,25,27,28,29,33,34]+[5,9,10,12,13,14,15,16]+[1,2,3,4,20]t

Table B9:CBRMDs for $v = 9i + 8$, $i > 3$(odd) $p_1 = 9$ and $p_2 = 8$, $\lambda' = 1$

	Set of Shifts
3	[6,7,8,46,47,48,49,50]+[18,19,20,40,41,42,43,44]+ [30,31,32,34,35,36,37,38]+[12,13,15,25,26,27,28,29]+ [9,10,14,16,17,21,22,23]+[1,2,3,4,5,11]t