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Entropy for Concomitants of k-Record Values in Morgenstern Family of Distributions

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ABSTRACT

In this paper, we obtain the expression for Shannon entropy of concomitants of k-record values arising from Morgenstern family of distributions. Based on this expression we derive the entropy of certain distributions belonging to Morgenstern family of distributions. Some properties for entropy of concomitants of k-record values arising from Morgenstern family of distributions are also discussed.

1. Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having a common cumulative distribution function (cdf) $F(x)$ which is absolutely continuous. An observation X_j is called an upper record if its value exceeds that of all preceeding observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. In an analogous way one can define lower record values also.

Interest in records has increased steadily over the years since its formulation by Chandler (1952). Record value data arise in a wide variety of practical situations. Examples include destructive stress testing, sporting and athletic events, meteorological analysis, oil and mining surveys, hydrology, seismology etc. For a detailed survey on the theory and application of record values see, Ahsanullah

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(2004), Arnold et al. (1998), Nevzorov (2001) and the references therein.
There are several situations where the second or third largest values are of special interest. For example, in actuarial science if the insurance claims of some non-life insurance is considered, then the usual record models is inadequate (see, Kamps, 1995). Also, in many practical situations occurrences of record values are very rare and the expected waiting time is infinite for every record after the first. These problems are avoided if we consider the model of k-record statistics introduced by Dziubdziela and Kopocinski (1976).
For a positive integer k, the upper k-record times $T_{n(k)}$ and the upper k-record values $U_{n(k)}$ are defined as follows:

$$T_{1(k)} = k, \ \ with \ \ probability\ 1$$

and, for $n > 1$

$$T_{n(k)} = min\{ j : j > T_{n-1(k)}, X_j > X_{T_{n-1(k)}-k+1:T_{n-1(k)}} \}, \tag{1}$$

where $X_{i:m}$ is the *i*-th order statistic in a random sample of size m. The sequence of upper k-record values are then defined by

$$U_{n(k)} = X_{T_{n(k)}-k+1:T_{n(k)}} \quad for\ n \geq 1.$$

In an analogous way, we can define the n th lower k-record times and the n th lower k-values $L_{n(k)}$, for $n \geq 1$. For $k = 1$, the usual classical records are recovered.
Let $(X_1, Y_1), (X_2, Y_2)...$ be sequence of bivariate random variables from a continuous distribution. Let $\{R_n, n \geq 1\}$ is the sequence of record values in the sequence of X's, then the Y which corresponds with the nth record will be called the concomitant of the nth record, denoted by $R_{[n]}$. The concomitants of record values arise in a wide variety of practical experiments such as industrial stress testing, lifetime experiments, meteorological analysis, sporting matches and some other experimental fields. For other important applications of record values and their concomitants see Arnold et al. (1998) and Ahsanullah (1995). Some properties of concomitants of record values were discussed in Houchens (1984), Nevzorov and Ahsanullah (2000). For some applications of concomitants of record values see, Chacko and Thomas (2006, 2008). For a detailed discussion on the distribution theory of concomitants of record values see, Arnold *et al*. (1998).
Let $(X_i, Y_i), i \geq 1$ be a sequence of independent bivariate random variables with

common absolutely continuous joint cumulative distribution function (cdf) $F(x,y),(x,y)\in R\times R$. Let $F_X(x)$ and $F_Y(y)$ be the marginal cdf's of X and Y respectively. Let $U_{n(k)}, n\geq 0$ be the sequence of upper k-record values (see, Arnold et al., 1998) in the sequence of X's. Then the Y variable associated with the X value which when quantified as the nth upper k-record value is called the concomitants of nth upper k-record value and is denoted by $U_{[n(k)]}$. An analogous definition deals with the concomitant of nth lower k-record values. If the variable of interest is difficult or expensive to measure but an auxiliary variable which can be measured very easily but the occurrence of record values on auxiliary variable is very rare then concomitants of k record values are very useful.

In modeling bivariate data, when the prior information is in the form of marginal distributions, it is of advantage to consider families of bivariate distributions with specified marginals. The Morgenstern family of distributions (MFD) discussed in Kotz et al. (2000) is characterized by the specified marginal distribution functions $F_X(x)$ and $F_Y(y)$ of random variables X and Y respectively and a parameter α. The cumulative distribution function of MFD is given by

$$F_{X,Y}(x,y)=F_X(x)F_Y(y)\{1+\alpha(1-F_X(x))(1-F_Y(y))\},\ -1\leq\alpha\leq 1, \qquad (2)$$

with corresponding probability density function (pdf)

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)\{1+\alpha(2F_X(x)-1)(2F_Y(y)-1)\},\ -1\leq\alpha\leq 1. \qquad (3)$$

where $f_X(x)$ and $f_Y(y)$ are the pdf's corresponding to the distribution functions $F_X(x)$ and $F_Y(y)$ respectively. The parameter α is known as the association parameter, the two random variables are independent when α is zero. For suitable choice of $F_X(x)$ and $F_Y(y)$, the MFD is primarily useful as an alternative to the standard bivariate normal distribution (see,Conway, 1983, p.31). Shucany et al. (1978) have used MFD as a model for screening variables in quality control problems.

Shannon's entropy of a continuous random variable X, with pdf $f_X(x)$, is given by

$$H(X)=\int_{-\infty}^{+\infty} f_X(x)lnf_X(x)dx. \qquad (4)$$

This is a mathematical measure of information which measures the average reduction of uncertainty of X. The information measure for record values have

been investigated by several authors, including Zahedi and Shakil (2006), Baratpour et al. (2007), and Madadi and Tata (2009). Amini and Ahmadi (2007) investigated the properties of Fisher information in the sequence of the first n records and their concomitants. Tahmasebi and Behboodian (2012) has studied on information properties for concomitants of order statistics in Farlie-Gumbel-Morgenstern (FGM) family. Recently Tahmasebi (2013) obtained some results on entropy for concomitants of record values in Farlie-Gumbel-Morgenstern (FGM) family.
The organization of this paper is as follows. In section 2, we derive the Shannon entropy for concomitants of nth upper and lower k-record values arising from MFD. In section 3, as an illustration we derive the expression of entropy for certain distributions belong to Morgenstern family of distributions and study its properties. In section 4, we present the Kullback-Leibler distance between the nth and mth upper k-record values in MFD.

2. Entropy for Concomitants of k-record values from MFD

Let (X_i, Y_i), i=1,2,3... be a sequence of independent observations drawn from a distribution with pdf (3). If $U_{[n(k)]}$ is the concomitant of the nth upper k-record value of the X sequence , then the pdf of $U_{[n(k)]}$ is given by (see, Chacko and Mary, 2013)

$$f_{[n(k)]}(y) = f_Y(y)\left[1+\alpha\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\{2F_Y(y)-1\}\right]. \quad (5)$$

In the following theorem we derive the expression for Shannon entropy for concomitants of nth upper k-record value $U_{[n(k)]}$.

Theorem 1 *Let* (X_i, Y_i), *i=1,2,... be a sequence of independent observations drawn from a distribution with pdf given in (3). If* $U_{[n(k)]}$ *is the concomitant of the nth upper k-record value on the* X *sequence of observations, then the Shannon entropy of* $U_{[n(k)]}$ *for* $n \geq k$ *and* $\alpha \neq 0$ *is given by*

$$H(U_{[n(k)]}) = H(Y)[1-\alpha_{(n,k)}] - 2\alpha_{(n,k)}\phi(f) + C_\alpha(n,k), \quad (6)$$

where $\alpha_{(n,k)} = [\alpha\{1-2(\frac{k}{k+1})^{n+1}\}]$, $\phi(f) = \int_0^1 u log f_Y(F_Y^{-1}(u))du$, $H(Y)$ is the entropy of the parent distribution of Y and $C_\alpha(n,k)$ is given by

$$C_\alpha(n,k) = \frac{1}{8\alpha_{(n,k)}}[(1-\alpha_{(n,k)})^2(2log(1-\alpha_{(n,k)})-1)-(1+\alpha_{(n,k)})^2(2log(1+\alpha_{(n,k)})-1)].$$

Proof. From the definition of Shannon entropy given in (4) we have

$$H(U_{[n(k)]}) = -\int_{-\infty}^{\infty} f_{[n(k)]}(y)logf_{[n(k)]}(y)dy$$

$$= -\int_{-\infty}^{\infty} f_{[n(k)]}(y)log\left[f_Y(y)[1+\alpha\{1-2\left(\frac{k}{k+1}\right)^{n+1}\}\{2F_Y(y)-1\}]\right]dy$$

$$= -\int_{-\infty}^{\infty} f_{[n(k)]}(y)[logf_Y(y)+log[1+\alpha\{1-2\left(\frac{k}{k+1}\right)^{n+1}\}\{2F_Y(y)-1\}]]dy$$

$$= -E_{[n(k)]}[logf_Y(y)] - E_{[n(k)]}log[1+\alpha\{1-2\left(\frac{k}{k+1}\right)^{n+1}\}\{2F_Y(y)-1\}]$$

$$= -E_{[n(k)]}[logf_Y(y)] - E_{[n(k)]}log[1+\alpha_{(n,k)}\{2F_Y(y)-1\}],$$

where $\alpha_{(n,k)} = [\alpha\{1-2(\frac{k}{k+1})^{n+1}\}]$.

We have

$$-E_{[n(k)]}[logf_Y(y)] = -\int_{-\infty}^{\infty}(logf_Y(y))\, f_{[n(k)]}(y)dy$$

$$= -\int_{\infty}^{\infty} logf_Y(y)\left[f_Y(y)\left(1+\alpha\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\{2F_Y(y)-1\}\right)\right]dy$$

$$= -\int_{-\infty}^{\infty}(logf_Y(y))\, f_Y(y)dy - \int_{-\infty}^{\infty}(logf_Y(y))\, f_Y(y)\alpha_{(n,k)}\{2F_Y(y)-1\}dy$$

$$= H(Y) - \alpha_{(n,k)}\int_{-\infty}^{\infty} f_Y(y)logf_Y(y)\{2F_Y(y)-1\}dy$$

$$= H(Y) - \alpha_{(n,k)}\int_{-\infty}^{\infty} 2F_Y(y)\, f_Y(y)logf_Y(y)dy + \alpha_{(n,k)}$$

$$\int_{-\infty}^{\infty} f_Y(y)logf_Y(y)dy$$

$$= H(Y) - \alpha_{(n,k)}H(Y) - 2\alpha_{(n,k)}\int_{-\infty}^{\infty} F_Y(y)\, f_Y(y)logf_Y(y)dy.$$

Put $u = F_Y(y)$ in the integral of the above equation, we get

$$-E_{[n(k)]}[logf_Y(y)] = H(Y) - \alpha_{(n,k)}H(Y) - 2\alpha_{(n,k)}\int_0^1 ulogf(F_Y^{-1}(u))du$$

$$= H(Y)[1-\alpha_{(n,k)}]-2\alpha_{(n,k)}\int_0^1 u log f(F_Y^{-1}(u))du.$$

Again

$$E_{[n(k)]} log[1+\alpha_{(n,k)}\{2F_Y(y)-1\}] = \int_{-\infty}^{\infty} log[1+\alpha_{(n,k)}\{2F_Y(y)-1\}]\, f_{[n(k)]}(y)dy.$$

Put

$$u = log[1+\alpha_{(n,k)}(2F_Y(y)-1)]$$

Then

$$e^u = [1+\alpha_{(n,k)}(2F_Y(y)-1)]$$

and

$$du = \frac{2}{e^u}\alpha_{(n,k)} f_Y(y)dy.$$

Thus

$$E_{[n(k)]}\{log\left[1+\alpha_{(n,k)}(2F_Y(y)-1)\right]\} = \int_{log(1-\alpha_{(n,k)})}^{log(1+\alpha_{(n,k)})} ue^{2u}\frac{1}{2\alpha_{(n,k)}}du$$

$$= \frac{1}{2\alpha_{(n,k)}}\int_{log(1-\alpha_{(n,k)})}^{log(1+\alpha_{(n,k)})} ue^{2u}du$$

$$= \frac{1}{2\alpha_{(n,k)}}\left\{\frac{(1+\alpha_{(n,k)})^2 log(1+\alpha_{(n,k)})}{2}\right.$$

$$\left.-\frac{(1-\alpha_{(n,k)})^2 log(1-\alpha_{(n,k)})}{2}-\frac{(1+\alpha_{(n,k)})^2}{4}+\frac{(1-\alpha_{(n,k)})^2}{4}\right\}$$

$$= \frac{1}{8\alpha_{(n,k)}}\left[(1+\alpha_{(n,k)})^2\left(2log(1+\alpha_{(n,k)})-1\right)\right.$$

$$\left.-(1-\alpha_{(n,k)})^2\left(2log(1-\alpha_{(n,k)})-1\right)\right].$$

Therefore we get

$$-E_{n(k)}\left[log\left(1+\alpha_{(n,k)}(2F_Y(y)-1)\right)\right] = C_\alpha(n,k),$$

where

$$C_\alpha(n,k) = \frac{1}{8\alpha_{(n,k)}}[(1-\alpha_{(n,k)})^2(2log(1-\alpha_{(n,k)})-1)-(1+\alpha_{(n,k)})^2(2log(1+\alpha_{(n,k)})-1)]. \tag{10}$$

Substituting (8) and (9) in (7), we have,

$$H(U_{[n(k)]}) = H(Y)\left[1-\alpha_{(n,k)}\right]-2\alpha_{(n,k)}\int_0^1 u log f(F_Y^{-1}(u))du + C_\alpha(n,k)$$

$$= H(Y)\left[1-\alpha_{(n,k)}\right]-2\alpha_{(n,k)}\phi(f)+C_\alpha(n,k),$$

where $C_\alpha(n,k)$ is given as in (10) and $\phi(f) = \int_0^1 ulogf(F_Y^{-1}(u))du$.

Theorem 2. *Let* (X_i, Y_i) *, i=1,2,3... be a sequence of independent observations drawn from (2). If* $L_{n(k)}$ *is the concomitant of nth lower k-record value on the X sequence of observations, then the Shannon entropy of* $L_{[n(k)]}$ *for* $n \geq k$ *and* $\alpha \neq 0$ *is given by*

$$H(L_{[n(k)]}) = H(Y)[1+\alpha_{(n,k)}] + 2\alpha_{(n,k)}\phi(f) + C_\alpha(n,k)$$

where $C_\alpha(n,k)$ is given as in (10) and $\phi(f) = \int_0^1 ulogf(F_Y^{-1}(u))du$.

Proof. The pdf of $L_{[n(k)]}$ is given by

$$f_{[n(k)]}(y) = f_Y(y)\left[1-\alpha\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\{2F_Y(y)-1\}\right].$$

From(5) and (12), it is clear that the pdf of concomitant of the nth upper k-record for a given α is same as that of the pdf of concomitant of the nth lower k-record for $-\alpha$. Hence the Shannon entropy of $L_{[n(k)]}$ is obtained by changing α to $-\alpha$ in (6). Since

$C_{-\alpha}(n,k) = C_\alpha(n,k)$ we get

$$H(L_{[n(k)]}) = H(Y)[1+\alpha_{(n,k)}] + 2\alpha_{(n,k)}\phi(f) + C_\alpha(n,k). \qquad (13)$$

Hence the theorem.

In the following theorem, we provide entropy bounds for concomitants of k-record values in Morgenstern family of distribution.

Theorem 3 *Let* $U_{[n(k)]}$ *be the concomitant of the* nth *upper k-record value in Morgenstern family of distribution. Then for* $n > k$ *, and* $\alpha \neq 0$ *, we have*

$$C_\alpha(n,k) + H(Y)\left[1-|\alpha|\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\right] \leq H(U_{[n(k)]}) \leq C_\alpha(n,k) + H(Y)\left[1+|\alpha|\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\right]$$

Proof. From theorem 2.1, we have

$$H(U_{[n(k)]}) = C_\alpha(n,k) + H(Y) + D_\alpha(n,k) \tag{14}$$

where

$$D_\alpha(n,k) = \alpha\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\int_{-\infty}^{\infty}[1-2F_Y(y)]f(y)logf(y)dy,$$

and $C_\alpha(n,k)$ is defined in (10). Since $-1 \le [1-2F_Y(y)] \le 1$, we have

$$-|\alpha|\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}H(Y) \le D_\alpha(n,k) \le |\alpha|\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}H(Y). \tag{15}$$

Thus by (14) and (15), the proof is clear.

3. Illustrations

In this section, we derive the Shannon entropy for concomitant of nth upper k-record value arising from certain distributions belong to Morgenstern family of distributions.

Example 3.1 Let $U_{[n(k)]}$ be the concomitant of nth upper k-record value arising from Morgenstern type bivariate uniform distribution (MTBUD) with cdf given by

$$F_{X,Y}(x,y) = xy\{1+\alpha(1-x)(1-y)\},$$
$$0 < x < 1, 0 < y < 1, -1 < \alpha < 1.$$

Then the pdf of $U_{[n(k)]}$ is given by

$$f_{[n(k)]}(y) = 1+\alpha\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}(2y-1), 0 < y < 1.$$

Then the entropy of $U_{[n(k)]}$ is given by

$$H_{[n(k)]}(Y) = C_\alpha(n,k)$$

where $C_\alpha(n,k)$ is given as in (10).

It is easy to check the following properties of $H_{[n(k)]}(Y)$.

1. $H_{[n(k)]}(Y) = C_\alpha(n,k) = C_{-\alpha}(n,k)$; $\forall n > k$ and $\alpha \ne 0$.
2. $H_{[n(k)]}(Y)$ is decreasing in α
3. $H_{[n(k)]}(Y)$ is decreasing in n.

We have evaluated numerical values of $H_{[n(k)]}(Y)$ for $\alpha = 0.25(0.25)\,0.75$, $k = 2(1)4$ and $n = 2(1)10$ and are given in table 1. From the table we can verify the above properties of entropy of nth k record value arising from MTBUD .

Example 3.2 Let $U_{[n(k)]}$ be the concomitant of nth upper k-record value arising from Morgenstern type bivariate exponential distribution (MTBED) with cdf given by (see, Kotz et al. 2000)

$$F_{X,Y}(x,y) = \left[1 - exp\left\{\frac{-x}{\theta_1}\right\}\right]\left[1 - exp\left\{\frac{-y}{\theta_2}\right\}\right]\left[1 + \alpha\, exp\left\{-\left(\frac{x}{\theta_1} + \frac{y}{\theta_2}\right)\right\}\right].$$

Then the pdf of $U_{[n(k)]}$ is given by

$$f_{[n(k)]}(y) = \frac{1}{\theta_2} exp\left\{\frac{-y}{\theta_2}\right\}\left[1 + \alpha\left\{1 - 2\left(\frac{k}{k+1}\right)^{n+1}\right\}\left(1 - 2exp\left\{\frac{-y}{\theta_2}\right\}\right)\right]; n \geq 0.$$

Then the entropy of $U_{[n(k)]}$ is given by

$$H_{[n(k)]}(Y) = 1 + \frac{\alpha_{(n,k)}}{2} + log\theta_2 + C_\alpha(n,k)$$

where $C_\alpha(n,k)$ is given as in (10) and $\alpha_{(n,k)} = \alpha\left(1 - 2\left(\frac{k}{k+1}\right)^{n+1}\right)$.

It is easy to check the following properties of $H_{[n(k)]}(Y)$.

1. $H_{[n(k)]}(Y)$ is monotone increasing in α, for $n > k$, $\forall \theta_2 > 0$.
2. $H_{[n(k)]}(Y)$ is an increasing concave function of θ_2, for $n > k$, $\alpha \neq 0$.
3. $H_{[n(k)]}(Y)$ is increasing in n for $0 < \alpha \leq 1$ and decreasing for $-1 \leq \alpha < 0$.
4. $H_{[n(k)]}(Y)$ is decreasing in k for $0 < \alpha \leq 1$ and increasing for $-1 \leq \alpha < 0$.

We have evaluated numerical values of $H_{[n(k)]}(Y)$ for $\alpha = -0.25(0.25)\,0.75(\alpha \neq 0)$, $k = 2(1)4$, $n = 2(1)10$ and $\theta = 0.5,2$ and are given in table 2 and table 3.

Example 3.3 Let (X_i, Y_i) ;i=1,2,3... be a sequence of independent observations drawn from Morgenstern type bivariate Pareto distribution (MTBPD) with cdf given by

$$F_{X,Y}(x,y) = (1-x^{-\lambda_1})(1-y^{-\lambda_2})[1+\alpha(x^{-\lambda_1})(y^{-\lambda_2})], x, y > 1; \lambda_1, \lambda_2 > 0.$$

Let $f_{[n(k)]}$ be the pdf of concomitant of nth upper k-record value and is given by

$$f_{[n(k)]}(y) = \lambda_2 y^{-\lambda_2 - 1}\left[1+\alpha\left\{1-2\left(\frac{k}{k+1}\right)^{n+1}\right\}\{1-2y^{-\lambda_2}\}\right]$$

Then the entropy is given by

$$H_{[n(k)]}(Y) = C_\alpha(n,k) - log\lambda_2 + \left(\frac{\lambda_2+1}{\lambda_2}\right)\left[1+\frac{\alpha_{(n,k)}}{2}\right]$$

where $C_\alpha(n,k)$ is given as in (10) and $\alpha_{(n,k)} = \alpha\left(1-2\left(\frac{k}{k+1}\right)^{n+1}\right)$.

It is easy to check the following properties of $H_{[n(k)]}(Y)$.

1. $H_{[n(k)]}(Y)$ is monotone decreasing in λ_2, for $n > k$, $\alpha \neq 0$.
2. $H_{[n(k)]}(Y)$ is monotone increasing in α , for $n > k$, $\forall \lambda_2 > 0$.
3. $H_{[n(k)]}(Y)$ is increasing in n for $0 < \alpha \leq 1$ and decreasing for $-1 \leq \alpha < 0$.
4. $H_{[n(k)]}(Y)$ is decreasing in k for $0 < \alpha \leq 1$ and increasing for $-1 \leq \alpha < 0$.

We have evaluated numerical values of $H_{[n(k)]}(Y)$ for α = –0.25(0.25) 0.75($\alpha \neq 0$) , k = 2(1)4 , n = 2(1)10 and $\lambda = 1,2$ and are given in table 4 and table 5.

4. Kullback-Leibler Distance

Kullback-Leibler distance is also called as $K-L$ Divergence or relative entropy. The Kullback-Leibler distance or divergence for two continuous random variables Z_1 and Z_2 with pdf f_1 and f_2 respectively, is given by, (see, Kullback and Leibler, 1951)

$$K(Z_1,Z_2) = \int_{-\infty}^{\infty} f_1(z) log\left(\frac{f_1(z)}{f_2(z)}\right)dz = E_1\left(log\frac{f_1(z)}{f_2(z)}\right),$$

where E_1 denotes the expectation with respect to f_1. A very essential property of $K(Z_1,Z_2)$ will be that the Kullback-Leibler distance is always non-negative and is equal to zero if and only if $f_1 = f_2$ almost everywhere. $K(Z_1,Z_2)$ is not symmetric and is additive for independent random events. The $K-L$ divergence remains well-defined for continuous distributions. Furthermore, it is invariant under one to one transformation and thereby Kullback-Leibler divergence between two location-scale models does not depend on the parameters of the model from where the divergence is measured. For more details on K-L divergence and its properties, one may refer Kullback and Leibler (1951) and Kullback (1959). Few authors like Ebrahimi et al. (2004) and Tahmasebi and Behboodian (2012) have considered K-L divergence information measure for order statistics and concomitants of order statistics respectively. In the following theorem, we consider the K-L distance for concomitants of nth and mth upper k-record value.

Theorem 4 *Let $U_{[m(k)]}$ and $U_{[n(k)]}$ be the concomitants of the mth and nth upper k-record values arising from Morgenstern type bivariate uniform distribution. Then the Kullback-Leibler distance between $U_{[m(k)]}$ and $U_{[n(k)]}$ is as follows:*

$$K(U_{[n(k)]},U_{[m(k)]}) = C_\alpha(n,k) - C_\alpha(m,k)\left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}}\right]$$

$$+\left\{\left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}}\right]-1\right\}\frac{1}{2\alpha_{(m,k)}}\{(1+\alpha_{(m,k)})log(1+\alpha_{(m,k)})$$

$$-(1-\alpha_{(m,k)})log(1-\alpha_{(m,k)})-2\alpha_{(m,k)}\}.$$

Proof. From the definition (17) and using the pdf given in (16), we have

$$K\left[U_{[n(k)]},U_{[m(k)]}\right]=\int_0^1 f_{[n(k)]}(y)log\,\frac{f_{[n(k)]}(y)}{f_{[m(k)]}(y)}\,dy$$

$$=\int_0^1 f_{[n(k)]}(y)logf_{[n(k)]}(y)-\int_0^1 f_{[n(k)]}(y)logf_{[m(k)]}(y)$$

$$=\int_0^1\left[1+\alpha_{(n,k)}(2y-1)\right]logf_{[n(k)]}(y)dy-$$

$$\int_0^1\left[1+\alpha_{(n,k)}(2y-1)\right]logf_{[m(k)]}(y)dy$$

$$=C_\alpha(n,k)-\int_0^1\left[1+\alpha_{(n,k)}(2y-1)\right]logf_{[m(k)]}(y)dy$$

$$\int_0^1 f_{[n(k)]}(y)logf_{[m(k)]}(y)dy=\int_0^1\left[1+\alpha_{(n,k)}(2y-1)\right]logf_{[m(k)]}(y)dy$$

$$=\int_0^1\left[1+\alpha_{(n,k)}(2y-1)\right]log\left[1+\alpha_{(m,k)}(2y-1)\right]dy$$

Put $u=log\left[1+\alpha_{(m,k)}(2y-1)\right]$.

Then

$$e^u=\left[1+\alpha_{(m,k)}(2y-1)\right]$$

and

$$dy=\frac{e^u}{2\alpha_{(m,k)}}du.$$

On substituting, we have,

$$\int_0^1 f_{[n(k)]}(y)logf_{[m(k)]}(y)dy=\int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})}\left\{1+\left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}}\right][e^u-1]\right\}\frac{ue^u}{2\alpha_{(m,k)}}du$$

$$=\int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})}\frac{ue^u}{2\alpha_{(m,k)}}du+\int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})}ue^u[e^u-1]\frac{1}{2\alpha_{(m,k)}}\left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}}\right]du$$

$$=I_1+I_2,$$

where

$$I_1 = \int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})} \frac{ue^u}{2\alpha_{(m,k)}} du$$

and

$$I_2 = \int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})} ue^u[e^u - 1] \frac{1}{2\alpha_{(m,k)}} \left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}} \right] du.$$

Then

$$I_1 = \int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})} \frac{ue^u}{2\alpha_{(m,k)}} du$$

$$= \frac{1}{2\alpha_{(m,k)}} \left\{ (1+\alpha_{(m,k)}) log(1+\alpha_{(m,k)}) - (1-\alpha_{(m,k)}) log(1-\alpha_{(m,k)}) - 2\alpha_{(m,k)} \right\}.$$

Then

$$I_2 = \int_{log(1-\alpha_{(m,k)})}^{log(1+\alpha_{(m,k)})} ue^u[e^u - 1] \frac{1}{2\alpha_{(m,k)}} \left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}} \right] du$$

$$= \left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}} \right] \left\{ C_\alpha(m,k) - \frac{1}{2\alpha_{(m,k)}} [(1+\alpha_{(m,k)}) log(1+\alpha_{(m,k)}) \right.$$

$$\left. - (1-\alpha_{(m,k)}) log(1-\alpha_{(m,k)}) - 2\alpha_{(m,k)}] \right\}.$$

On substituting I_1 and I_2 in (20), we get,

$$\int_0^1 f_{[n(k)]}(z) log f_{[m(k)]}(z) dz = C_\alpha(m,k) \left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}} \right] + \left\{ \left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}} \right] - 1 \right\}$$

$$\frac{1}{2\alpha_{(m,k)}} \{ (1+\alpha_{(m,k)}) log(1+\alpha_{(m,k)}) - (1-\alpha_{(m,k)})$$

$$log(1-\alpha_{(m,k)})-2\alpha_{(m,k)}\}.$$

On substituting (21) in (19), we get,

$$K\left[U_{[n(k)]},U_{[m(k)]}\right]=C_\alpha(n,k)-C_\alpha(m,k)\left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}}\right]+\left\{\left[\frac{1-2\left(\frac{k}{k+1}\right)^{n+1}}{1-2\left(\frac{k}{k+1}\right)^{m+1}}\right]-1\right\}$$

$$\frac{1}{2\alpha_{(m,k)}}\{(1+\alpha_{(m,k)})log(1+\alpha_{(m,k)})-(1-\alpha_{(m,k)})log(1-\alpha_{(m,k)})-2\alpha_{(m,k)}\}.$$

Hence the theorem.

Table 1: Entropy of concomitant of n th upper k record arising from MTBUD

k	n	α		
		0.25	0.5	0.75
2	2	-0.001731	-0.006945	-0.015710
	3	-0.003821	-0.015391	-0.035060
	4	-0.005672	-0.022928	-0.052579
	5	-0.007110	-0.028826	-0.066474
	6	-0.008161	-0.033154	-0.076789
	7	-0.008902	-0.036220	-0.084159
	8	-0.009414	-0.038345	-0.089300
	9	-0.009764	-0.039798	-0.092833
	10	-0.010001	-0.040783	-0.095235
3	2	-0.000254	-0.001018	-0.002292
	3	-0.001406	-0.005637	-0.012738
	4	-0.002880	-0.011582	-0.026299
	5	-0.004332	-0.017468	-0.039861
	6	-0.005616	-0.022702	-0.052049
	7	-0.006690	-0.027099	-0.062386
	8	-0.007557	-0.030666	-0.070846
	9	-0.008243	-0.033495	-0.077605
	10	-0.008778	-0.035705	-0.082916

4	2	-0.000006	-0.000024	-0.000054
	3	-0.000341	-0.001363	-0.003070
	4	-0.001238	-0.004964	-0.011211
	5	-0.002361	-0.009483	-0.021496
	6	-0.003518	-0.014166	-0.032234
	7	-0.004612	-0.018606	-0.042500
	8	-0.005594	-0.022610	-0.051833
	9	-0.006448	-0.026107	-0.060047
	10	-0.007176	-0.029096	-0.067114

Table 2: Entropy of concomitant of n th upper k record arising from MTBED when $\theta_2 = 0.5$

k	n	α					
		0.25	0.5	0.75	-0.25	-0.5	-0.75
2	2	0.356048	0.40176	0.443921	0.254196	0.198056	0.138365
	3	0.378649	0.442696	0.498645	0.227415	0.140227	0.044941
	4	0.393259	0.468081	0.530508	0.209103	0.099768	-0.02196
	5	0.402795	0.484131	0.549536	0.19669	0.071923	-0.06878
	6	0.40906	0.494434	0.561168	0.188324	0.052962	-0.10104
	7	0.413196	0.501123	0.56843	0.182705	0.040142	-0.12304
	8	0.415935	0.505502	0.573043	0.178941	0.031514	-0.13794
	9	0.417753	0.508384	0.576014	0.176424	0.025726	-0.14797
	10	0.418962	0.51029	0.577947	0.174742	0.021851	-0.15471
3	2	0.32613	0.344897	0.363155	0.287067	0.266772	0.245967
	3	0.351346	0.393013	0.43181	0.259549	0.209419	0.15642
	4	0.369646	0.426618	0.477575	0.238299	0.163923	0.083532
	5	0.383026	0.450396	0.508508	0.222015	0.128374	0.025476
	6	0.392866	0.467409	0.529691	0.209608	0.100893	-0.02008
	7	0.400135	0.479698	0.544382	0.200191	0.079811	-0.05545
	8	0.405524	0.488645	0.554693	0.193067	0.063729	-0.08268
	9	0.409531	0.495201	0.562013	0.187688	0.051514	-0.10352

	10	0.412516	0.500031	0.567261	0.183634	0.042266	-0.11939
	2	0.303847	0.300829	0.297799	0.309847	0.312829	0.315799
	3	0.329112	0.35069	0.371583	0.283912	0.26029	0.235983
	4	0.348695	0.388049	0.424882	0.262535	0.215729	0.166402
	5	0.363956	0.416297	0.463749	0.245028	0.178441	0.106965
	6	0.375906	0.43783	0.492333	0.230763	0.147545	0.056905
	7	0.385298	0.454361	0.513523	0.219184	0.122133	0.015182
	8	0.392705	0.467134	0.529356	0.209814	0.101352	-0.01932
	9	0.398561	0.477058	0.541275	0.202248	0.084433	-0.04766
	10	0.403202	0.484807	0.550314	0.196152	0.070707	-0.07084

Table 3: Entropy of concomitant of n th upper k record arising from MTBED when $\theta_2 = 2$

k	n	α					
		0.25	0.5	0.75	-0.25	-0.5	-0.75
2	2	1.742342	1.788054	1.830215	1.64049	1.58435	1.524659
	3	1.764944	1.828991	1.884939	1.613709	1.526521	1.431236
	4	1.779554	1.854375	1.916803	1.595397	1.486062	1.364334
	5	1.789089	1.870426	1.93583	1.582985	1.458217	1.317517
	6	1.795354	1.880729	1.947463	1.574618	1.439257	1.285254
	7	1.79949	1.887418	1.954725	1.569	1.426436	1.263252
	8	1.80223	1.891796	1.959338	1.565236	1.417808	1.248356
	9	1.804048	1.894679	1.962309	1.562718	1.41202	1.238321
	10	1.805256	1.896584	1.964242	1.561037	1.408145	1.231583
3	2	1.712424	1.731192	1.749449	1.673362	1.653067	1.632261
	3	1.73764	1.779307	1.818104	1.645843	1.595713	1.542714
	4	1.755941	1.812912	1.863869	1.624593	1.550217	1.469826
	5	1.769321	1.83669	1.894802	1.60831	1.514669	1.41177
	6	1.77916	1.853703	1.915985	1.595902	1.487187	1.366211
	7	1.786429	1.865992	1.930676	1.586486	1.466105	1.330845
	8	1.791819	1.874939	1.940988	1.579361	1.450024	1.303615
	9	1.795825	1.881495	1.948307	1.573982	1.437809	1.282777

	10	1.798811	1.886325	1.953555	1.569928	1.42856	1.266908
4	2	1.690141	1.687123	1.684093	1.696141	1.699123	1.702093
	3	1.715407	1.736984	1.757877	1.670207	1.646584	1.622277
	4	1.734989	1.774343	1.811176	1.648829	1.602023	1.552696
	5	1.750251	1.802592	1.850043	1.631323	1.564736	1.493259
	6	1.7622	1.824124	1.878627	1.617057	1.533839	1.4432
	7	1.771592	1.840655	1.899818	1.605478	1.508428	1.401476
	8	1.778999	1.853428	1.91565	1.596108	1.487646	1.366977
	9	1.784855	1.863353	1.92757	1.588543	1.470727	1.338631
	10	1.789496	1.871102	1.936608	1.582446	1.457001	1.315457

Table 4: Entropy of concomitant of n th upper k record arising from MTBPD when $\lambda_2 = 1$

k	n	α					
		0.25	0.5	0.75	-0.25	-0.5	-0.75
	2	2.100121	2.196759	2.289846	1.896417	1.789351	1.678734
	3	2.147414	2.287078	2.418644	1.844945	1.68214	1.511237
	4	2.178485	2.345384	2.49989	1.810172	1.608759	1.394952
	5	2.198994	2.383383	2.551839	1.786785	1.558966	1.315214
	6	2.212575	2.408318	2.58542	1.771103	1.525373	1.261003
	7	2.221589	2.424761	2.607314	1.760607	1.502798	1.224369
	8	2.227579	2.435643	2.621681	1.753592	1.487667	1.199718
	9	2.231565	2.442861	2.631155	1.748907	1.477544	1.18318
	10	2.234219	2.447656	2.637424	1.74578	1.470778	1.172107
	2	2.038808	2.077107	2.114896	1.960683	1.920857	1.880521
	3	2.090391	2.177957	2.262653	1.906797	1.810769	1.711871
	4	2.128467	2.251113	2.367744	1.865772	1.725722	1.579658
	5	2.156679	2.304554	2.443171	1.834657	1.660511	1.477107
	6	2.177642	2.343814	2.497725	1.811126	1.610782	1.398177
	7	2.193254	2.372788	2.537444	1.793367	1.573014	1.337783

	8	2.2049	2.394249	2.566527	1.779985	1.544419	1.291781
	9	2.2136	2.410191	2.587925	1.769913	1.522818	1.256865
	10	2.220105	2.42206	2.603732	1.76234	1.50653	1.230437
	2	1.993994	1.987976	1.981946	2.005994	2.011976	2.017946
	3	2.044859	2.089037	2.13253	1.954459	1.908237	1.86133
	4	2.084922	2.167356	2.247269	1.912602	1.822716	1.730309
	5	2.116567	2.228373	2.335288	1.878711	1.752661	1.62172
	6	2.141624	2.276119	2.403193	1.851339	1.69555	1.532339
	7	2.161502	2.313622	2.455841	1.829274	1.649167	1.459158
	8	2.177297	2.343172	2.49684	1.811515	1.611608	1.399493
	9	2.189865	2.366519	2.528892	1.797239	1.581267	1.351014
	10	2.199874	2.385005	2.554036	1.785774	1.556803	1.311735

Table 5: Entropy of concomitant of nth upper k record arising from MTBPD when $\lambda_2 = 2$

k	n	α					
		0.25	0.5	0.75	-0.25	-0.5	-0.75
2	2	0.881511	0.952686	1.020309	0.728733	0.64713	0.561976
	3	0.916458	1.018313	1.112071	0.689606	0.56461	0.431515
	4	0.939298	1.060159	1.168626	0.663064	0.50769	0.339922
	5	0.954321	1.087184	1.204114	0.645164	0.468871	0.276645
	6	0.964244	1.104803	1.22672	0.63314	0.442594	0.233408
	7	0.970819	1.116369	1.241298	0.625083	0.424896	0.20409
	8	0.975184	1.123999	1.250789	0.619693	0.413017	0.184316
	9	0.978086	1.129049	1.257011	0.616092	0.405061	0.17103
	10	0.980017	1.132399	1.261112	0.613687	0.399741	0.162124
3	2	0.835895	0.864429	0.892451	0.777302	0.747241	0.71667
	3	0.874295	0.938911	1.000658	0.7366	0.663521	0.587572
	4	0.902483	0.992292	1.076086	0.705462	0.598249	0.485021
	5	0.923279	1.030901	1.129266	0.681763	0.547869	0.404717
	6	0.93868	1.059038	1.167134	0.663793	0.509264	0.342473
	7	0.950121	1.079669	1.194339	0.650205	0.479839	0.294593

	8	0.958639	1.094873	1.214037	0.639952	0.4575	0.257977
	9	0.964992	1.106123	1.228395	0.632227	0.440593	0.230101
	10	0.969737	1.114472	1.238923	0.626413	0.427824	0.208952
4	2	0.802347	0.797829	0.793299	0.811347	0.815829	0.820299
	3	0.840412	0.87329	0.905483	0.772612	0.73769	0.702083
	4	0.870235	0.931129	0.989502	0.740995	0.672649	0.601782
	5	0.893688	0.975761	1.052945	0.715296	0.618977	0.517769
	6	0.912191	1.010401	1.101189	0.694478	0.574974	0.448049
	7	0.926826	1.037418	1.138109	0.677656	0.539076	0.390596
	8	0.938427	1.05858	1.166524	0.664091	0.509906	0.343514
	9	0.947639	1.075215	1.18851	0.65317	0.486276	0.305102
	10	0.954965	1.088332	1.205602	0.644389	0.467181	0.273875

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