ISSN 0971-0388

Aligarh Journal of Statistics
Vol. 36 (2016), 63-78

On Generalized (k) Record Values From Pareto Distribution

Jerin Paul[1] and P. Yageen Thomas[2]

[Received on July, 2014. Revised on November, 2016]

ABSTRACT

In this paper we study the Generalized (k) record values arising from Pareto distribution. Expressions for the moments and product moments of generalized upper (k) record values arising from Pareto distribution are derived. Some properties of generalized (k) record values which characterize the Pareto distribution have been established. Also some interesting distributional properties of generalized upper (k) record values arising from Pareto distribution are established. The scale and shape parameters of Pareto distribution are estimated based on available record value data. Prediction of future records using Best Linear Unbiased Predictors has been studied.

1. INTRODUCTION

Record values can be viewed as successive extremes occurring in a sequence of random variables. Record data arise in a wide variety of practical situations, which include industrial stress testing, meteorological studies, hydrology, seismology, sporting and athletic events and oil and mining investigations. The statistical study of records was first made in a paper by Chandler (1952). For more details on the applications of record values, see Arnold *et al.* (1998), Nevzorov (2001) and Ahsanullah (1998).

---

[1]*Address for Correspondence:* Jerin Paul, Department of Statistics, University of Kerala, Kariavattom, Trivandrum -695581. India. E-mail: jerinstat@gmail.com and P. Yageen Thomas[2], Department of Statistics, University of Kerala Trivandrum, India.

A difficulty that one encounters in dealing with statistical inference problems based on record values is about their limited occurrence, as the expected waiting time of occurrence of every record after the first is infinite. However one may observe that generally the kth record values as introduced by Dziubdziela and Kopocinski (1976) occur more frequently than those of the classical records. Further the sequence of kth record values for $k > 1$ is free from the inclusion of outliers occurring in the data. For a positive integer $k \geq 1$, the sequence of upper kth record times $\{T_{U(n,k)}, n \geq 1\}$ is defined as (see, Arnold *et al.* 1998, p. 43):

$$T_{U(1,k)} = k, \text{ with probability 1}$$

and, for $n \geq 1$

$$T_{U(n+1,k)} = min\left\{j: j > T_{U(n,k)}, X_j > X_{T_{U(n,k)}-k+1:T_{U(n,k)}}\right\},$$

where $X_{i:m}$ denotes the ith order statistic in a sample of size m. Now if we write

$$X_{U(n,k)} = X_{T_{U(n,k)}-k+1:T_{U(n,k)}}, \text{ for } n = 1,2, \ldots$$

then $\{X_{U(n,k)}\}$ is known as the sequence of the kth upper record values. In an analogous way, one may define the kth lower record times and kth lower record values. Now from Arnold *et al.* (1998, p. 81) the pdf of $X_{U(n,k)}$ is given by

$$f_{X_{U(n,k)}}(x) = \frac{k^n}{\Gamma(n)}[-\log\{1 - F(x)\}]^{n-1}[1 - F(x)]^{k-1}f(x), \tag{1.1}$$

where $-\infty < x < \infty$. The kth member of the sequence of the classical record values is also called as kth record value. This contradicts with the kth record values as defined in Dziubdziela and Kopocinski (1976). Pointing out this conflict in the usage of kth record values and due to the reason that when $k = 1$ is used in the kth record values defined by Dziubdziela and Kopocinski (1976) the classical record values get generated, Minimol and Thomas (2013, 2014) have called the kth record values as defined in Dziubdziela and Kopocinski (1976) as the generalized record values. Agreeing with the contention of Minimol and Thomas (2013, 2014), we also call the kth record values of Dziubdziela and Kopocinski (1976) as generalized (k) record values all through this paper.

Pareto (1897) introduced the celebrated Pareto distribution (PD) which is found suitable for modelling income data of individuals whose income exceeded a given level x_0. Also the PD is widely used in modelling heavy-tailed distributions (see, Arnold, 2008). PD can be used to explain many socio-economic and other naturally occurring empirical phenomenas. Pareto models are found to be suitable to study city population sizes, stock price fluctuations, size of firms, personal incomes, and error clustering in communication circuits etc. and for details see, Johnson *et al.* (1994) and Arnold (1985).

A continuous random variable Z is said to have a Pareto distribution with shape parameter c and scale parameter σ if its cumulative distribution function (cdf) $F(z;\ \sigma, c)$ is given by

$$F(z;\ \sigma, c) = \begin{cases} 1-\left(\frac{z}{\sigma}\right)^{-c}, & z \geq \sigma, \sigma > 0, c > 0 \\ 0, & \text{otherwise.} \end{cases} \qquad (1.2)$$

The pdf corresponding to the cdf given in (1.2) is

$$f(z;\ \sigma, c) = \frac{c}{\sigma}\left(\frac{z}{\sigma}\right)^{-c-1}, \quad z \geq \sigma, \sigma > 0, c > 0. \qquad (1.3)$$

Also the standard form of the PD is given by the pdf

$$f_0(x) = cx^{-c-1}, \qquad x \geq 1, c > 0. \qquad (1.4)$$

The PD and its properties based on record values were extensively studied by many authors and such results are available in Balakrishnan and Ahasanullah (1994), Sulthan and Moshref (2000), Abd-El-Hakim and Sulthan (2004), Dallas (1982), Raqab and Awad (2000), Wu and Lee (2001), Chang (2009), Lee and Chang (2005), Lee and Lim (2009, 2011) and Ahasanullah and Shakil (2012).
In Section 2 of this paper we derive the exact expressions for the means, variances and covariances of generalized upper (k) record values (GURVs) arising from PD and also computed their numerical values. Further we have identified certain properties of GURVs which characterize the PD and those characterization theorems are given in Section 3. In Section 4 we have described some interesting distributional aspects of GURV arising from PD and explained some of its implications. Section 5 deals with the problem of (i) estimation of σ when c is known, (ii) estimation of σ and c when both are unknown. In Section 6 we consider the problem of prediction of a future GURV using an appropriate Best Linear Unbiased Predictor (BLUP) based on initial GURVs arising from a PD. In the last Section of this paper, we consider a numerical example through which we have illustrated the method of estimation of the parameters of PD and the method of predicting the next immediate future GURV by using the available GURVs.

2. GURV ARISING FROM STANDARD PARETO DISTRIBUTION

Let $\{X_{U(i,k)}\}$ be the sequence of GURVs arising from standard PD defined in (1.4). Then the pdf of $X_{U(n,k)}$ is given by

$$f_{X_{U(n,k)}}(x) = \frac{(kc)^n}{\Gamma(n)}(\ln(x))^{n-1}x^{-kc-1}, n = 1,2,\ldots, k = 1,2,\ldots, c > 0. \quad (2.1)$$

The joint pdf of $X_{U(m,k)}$ and $X_{U(n,k)}$ for $m < n$ is given by

$$f_{X_{U(m,k)},X_{U(n,k)}}(x,y) = \frac{(kc)^n}{\Gamma(m)\Gamma(n-m)} (\ln(x))^{m-1} \times (\ln(y) - \ln(x))^{n-m-1} x^{-1} y^{-kc-1}, \quad (2.2)$$

where $1 \leq x < y < \infty$. Now it is straight forward to obtain the following

$$\alpha_n^{(k)} = E\left(X_{U(n,k)}^i\right) = \left(\frac{kc}{kc-i}\right)^n, \text{provided } kc > i, \quad (2.3)$$

$$\alpha_{n,n}^{(k)} = Var\left(X_{U(n,k)}\right) = Cov\left(X_{U(n,k)}, X_{U(n,k)}\right)$$

$$= \left(\frac{kc}{kc-2}\right)^n - \left(\frac{kc}{kc-1}\right)^{2n}, \quad (2.4)$$

provided $kc > 2$, and

$$\alpha_{m,n}^{(k)} = Cov\left(X_{U(m,k)}, X_{U(n,k)}\right) = \frac{(kc)^n}{(kc-1)^{n-m}(kc-2)^m} - \left(\frac{kc}{kc-1}\right)^{n+m}, \quad (2.5)$$

provided $kc > 2$. We have computed the numerical values of the means and variances of GURV arising from the standard Pareto distribution for various values of $k = 2(1)5, n = 1(1)5$ and for $c = 2.5(0.5)5$ and are given in Table 1.

Table 1: Expected Values and Variances of GURVs arising from Standard PD

k	c	$\alpha_1^{(k)}$	$\alpha_{1,1}^{(k)}$	$\alpha_2^{(k)}$	$\alpha_{2,2}^{(k)}$	$\alpha_3^{(k)}$
2	2.5	1.25000	0.10417	1.56250	0.33637	1.95313
	3	1.20000	0.06000	1.44000	0.17640	1.72800
	3.5	1.16667	0.03889	1.36111	0.10738	1.58796
	4	1.14286	0.02721	1.30612	0.07182	1.49271
	4.5	1.12500	0.02009	1.26563	0.05125	1.42383
	5	1.11111	0.01543	1.23457	0.03834	1.37174
3	2.5	1.15385	0.03228	1.33136	0.08698	1.53619
	3	1.12500	0.02009	1.26563	0.05125	1.42383
	3.5	1.10526	0.01369	1.22161	0.03363	1.35020
	4	1.09091	0.00992	1.19008	0.02370	1.29827
	4.5	1.08000	0.00751	1.16640	0.01758	1.25971
	5	1.07143	0.00589	1.14796	0.01355	1.22996
4	2.5	1.11111	0.01543	1.23457	0.03834	1.37174
	3	1.09091	0.00992	1.19008	0.02370	1.29827
	3.5	1.07692	0.00690	1.15976	0.01606	1.24898
	4	1.06667	0.00508	1.13778	0.01158	1.21363

	4.5	1.05882	0.00389	1.12111	0.00874	1.18705
	5	1.05263	0.00308	1.10803	0.00683	1.16635
5	2.5	1.08696	0.00900	1.18147	0.02135	1.28421
	3	1.07143	0.00589	1.14796	0.01355	1.22996
	3.5	1.06061	0.00415	1.12489	0.00935	1.19306
	4	1.05263	0.00308	1.10803	0.00683	1.16635
	4.5	1.04651	0.00237	1.09519	0.00521	1.14613
	5	1.04167	0.00189	1.08507	0.00410	1.13028

3. CHARACTERIZATIONS OF PARETO DISTRIBUTION BASED ON GURV

Now we establish certain properties of GURVs which characterize the Pareto distribution.

Theorem 3.1 Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)}}{X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Proof.

The joint pdf of $X_{U(n,k)}$ and $X_{U(n+1,k)}$, is given by

$$f_{X_{U(n,k)},X_{U(n+1,k)}}(x,y) = k^{n+1}\frac{\{-\ln[\bar{F}(x)]\}^{n-1}}{\Gamma(n)}\frac{f(x)f(y)}{\bar{F}(x)}[\bar{F}(x)]^{k-1}, \quad (3.1)$$

where $\bar{F}(x) = 1 - F(x), 1 \leq x < y < \infty$. If $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, then the joint pdf $f_{X_{U(n,k)},X_{U(n+1,k)}}(x,y)$ of $X_{U(n,k)}$ and $X_{U(n+1,k)}$ is given by

$$f_{X_{U(n,k)},X_{U(n+1,k)}}(x,y) = \frac{(kc)^{n+1}}{\Gamma(n)}(\ln(x))^{n-1}x^{-1}y^{-kc-1}. \quad (3.2)$$

Consider the transformations $T = \frac{X_{U(n+1,k)}}{X_{U(n,k)}}$ and $Z = X_{U(n,k)}$. Then the Jacobian of the transformation is $|J| = z$. Thus we can write the joint pdf $f_{T,Z}(t,z)$ of T and Z as

$$\begin{aligned} f_{T,Z}(t,z) &= \frac{(kc)^{n+1}}{\Gamma(n)}(\ln(z))^{n-1}z^{-1}(tz)^{-kc-1}z \\ &= \frac{(kc)^{n+1}}{\Gamma(n)}(\ln(z))^{n-1}(tz)^{-kc-1}, t > 1, z \geq 1 \text{ and } c > 0. \quad (3.3) \end{aligned}$$

The marginal pdf of T is given by

$$f_T\,(t) = \frac{(kc)^{n+1}}{\Gamma(n)} t^{-kc-1} \int_1^{\infty} (\ln(z))^{n-1} z^{-kc-1} dz$$

$$= kc\, t^{-kc-1}, t > 1 \text{ and } c > 0. \qquad (3.4)$$

Also, the pdf $f_Z\,(z)$ of Z is given by

$$f_Z\,(z) = \frac{(kc)^n}{\Gamma(n)} (\ln(z))^{n-1} z^{-kc-1}, \;\; z \geq 1 \text{ and } c > 0. \qquad (3.5)$$

From (3.3), (3.4) and (3.5) we obtain $f_{T,Z}\,(t,z) = f_T\,(t) f_Z\,(z)$. Hence T and Z are independently distributed.

Now we prove the sufficient part of the Theorem. Let $T = \frac{X_{U(n+1,k)}}{X_{U(n,k)}}$ and $Z = X_{U(n,k)}$ be distributed independently. It follows that the Jacobian of the transformation is $|J| = z$. Thus we can write the joint pdf $f_{T,Z}\,(t,z)$ of T and Z as

$$f_{T,Z}\,(t,z) \; = \; k^{n+1} \frac{\{-\ln[\bar{F}(z)]\}^{n-1}}{\Gamma(n)} \frac{f(z) f(tz)}{\bar{F}(z)} [\bar{F}(tz)]^{k-1}, \qquad (3.6)$$

where $t > 1$ and $z \geq 1$. The pdf $f_Z\,(z)$ of Z is given by

$$f_Z\,(z) = k^n \frac{\{-\ln[\bar{F}(z)]\}^{n-1}}{\Gamma(n)} f(z) [\bar{F}(z)]^{k-1}, \;\; z \geq 1. \qquad (3.7)$$

Since T and Z are independent, we get the pdf $f_T\,(t)$ of T from (3.6) and (3.7) as

$$f_T\,(t) = k \frac{f(tz)}{\bar{F}(z)} \left[\frac{\bar{F}(tz)}{\bar{F}(z)}\right]^{k-1} z$$

$$= \frac{k}{\left(\bar{F}(z)\right)^k} [\bar{F}(tz)]^{k-1} f(tz) z.$$

Integrating the above expression with respect to t from τ to ∞, and simplifying we get

$$1 - F_T(\tau) = \frac{[1 - F(\tau z)]^k}{[1 - F(z)]^k}. \qquad (3.8)$$

The left side of (3.8) being the survival function of the random variable T, the right side should be independent of Z and is so only if,

$$\frac{[1 - F(\tau z)]^k}{[1 - F(z)]^k} = G(\tau),$$

and hence we have

$$[1 - F(\tau z)]^k = G(\tau)[1 - F(z)]^k,$$

where $G(\tau)$ is a function of τ alone. Since $1 \leq z < \infty$, on putting $z = 1$, in the above equation we get

$$[1 - F(\tau)]^k = G(\tau).$$

Hence we have

$$[1 - F(\tau z)]^k = [1 - F(\tau)]^k [1 - F(z)]^k. \quad (3.9)$$

The above equation is true for all $\tau > 1$ and $z \geq 1$. From Aczel (1966) we observe that, the only continuous solution of (3.9) with the boundary condition

$\bar{F}(1) = 1$ is $\bar{F}(x) = x^{-c}$, for all $x \geq 1$ and $c > 0$. Thus we have $F(x) = 1 - x^{-c}$. This completes the proof.

One can also state some more results whose proofs are just similar to the proof of Theorem 3.1 and hence we state those results without proof in the following corollaries.

Corollary 3.1 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n,k)}}{X_{U(n+1,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.2 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)}}{X_{U(n+1,k)} + X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.3 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)} + X_{U(n,k)}}{X_{U(n+1,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.4 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n,k)}}{X_{U(n+1,k)} + X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.5 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)} + X_{U(n,k)}}{X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.6 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let

$\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)}}{X_{U(n+1,k)} - X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.7 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let

$\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)} - X_{U(n,k)}}{X_{U(n+1,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.8 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n,k)}}{X_{U(n+1,k)} - X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.9 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)} - X_{U(n,k)}}{X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Corollary 3.9 Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURVs generated from the sequence $\{X_n\}$. Then $F(x) = 1 - x^{-c}$ for all $x \geq 1$, $c > 0$, if and only if $\frac{X_{U(n+1,k)} + X_{U(n,k)}}{X_{U(n+1,k)} - X_{U(n,k)}}$ and $X_{U(n,k)}$ are independently distributed.

Note 3.1 It is well known that independence of two variables implies non-correlation, where as non-correlation need not imply independence. Thus as an implication of the Theorem 3.1, if we have several data sets from a population, for any fixed n (number of records) then we may plot the points (T, Z) (where T and Z are variables as defined in Theorems 3.1) corresponding to each data set in different graph sheets. If the points in the graphs corresponding to (T, Z) are scattered all over the region without any pattern then it gives some clue to make further search to confirm if Pareto distribution could be taken as a suitable distribution of the population random variable.

4. SOME DISTRIBUTIONAL ASPECTS OF GURV ARISING FROM STANDARD PARETO DISTRIBUTION

In this section we derive some interesting properties of GURVs arising from standard PD.

Theorem 4.1 Let $r_1, r_2, \ldots, r_i$ be positive integers such that $1 \le r_1 < r_2 < \cdots < r_i$ and let $X_{U(r_1,k)}, X_{U(r_2,k)}, \ldots, X_{U(r_i,k)}$ be the corresponding GURVs arising from (1.4). Then the random variables $V_1 = X_{U(r_1,k)}, V_2 = \frac{X_{U(r_2,k)}}{X_{U(r_1,k)}},\ V_3 = \frac{X_{U(r_3,k)}}{X_{U(r_2,k)}}, \ldots, V_i = \frac{X_{U(r_i,k)}}{X_{U(r_{i-1},k)}}$ are all statistically

independent. Further the distribution of V_1, is given by the pdf $g_1(v_1) = (kc)^{r_1} \frac{[\ln (v_1)]^{r_1-1}}{\Gamma(r_1)} {v_1}^{-kc-1}, v_1 > 1$ and V_j's are distributed with pdf $h_j(v_j) = (kc)^{r_j - r_{j-1}} \frac{[\ln (v_j)]^{r_j - r_{j-1}-1}}{\Gamma(r_j - r_{j-1})} v_j^{-kc-1}, v_j > 1, j = 2,3, \ldots, i.$

Proof.

The joint distribution of $X_{U(r_1,k)}, X_{U(r_2,k)}, \ldots, X_{U(r_i,k)}$ is given by (see, Paul and Thomas, 2015)

$$f_{r_1,r_2,\ldots,r_i}(x_{r_1}, x_{r_2}, \ldots, x_{r_i}) = \frac{(kc)^{r_i}}{\Gamma(r_1)\Gamma(r_2-r_1)\Gamma(r_i - r_{i-1})} \times [\ln(x_{r_1})]^{r_1-1} \left[\ln\left(\frac{x_{r_2}}{x_{r_1}}\right)\right]^{r_2-r_1-1} \ldots \left[\ln\left(\frac{x_{r_i}}{x_{r_{i-1}}}\right)\right]^{r_i - r_{i-1}-1} \times (x_{r_1}\, x_{r_2} \ldots x_{r_i})^{-1} (x_{r_i})^{-kc}, \quad (4.1)$$

where $1 < x_{r_1} < x_{r_2} < \ldots < x_{r_i} < \infty$. If we put $v_1 = x_{r_1}, v_2 = \frac{x_{r_2}}{x_{r_1}}, v_3 = \frac{x_{r_3}}{x_{r_2}}, \ldots, v_{i-1} = \frac{x_{r_{i-1}}}{x_{r_{i-2}}}$ and $v_i = \frac{x_{r_i}}{x_{r_{i-1}}}$ then we have

$$x_{r_1} = v_1, x_{r_2} = v_1 v_2, \ldots, x_{r_{i-1}} = v_1 v_2 v_3 \ldots v_{i-1}, x_{r_i} = v_1 v_2 v_3 \ldots v_i. \quad (4.2)$$

The Jacobian of the transformation is given by

$$|J| = v_{i-1} v_{i-2}^2 \ldots v_2^{i-2} v_1^{i-1} \quad (4.3)$$

Using (4.2) in (4.1) and multiplying it by $|J|$ and simplifying we get

$$f(v_1, v_2, \ldots, v_i) = (kc)^{r_i} \frac{[\ln(v_1)]^{r_1-1}}{\Gamma(r_1)} \frac{[\ln(v_2)]^{r_2-r_1-1}}{\Gamma(r_2-r_1)} \frac{[\ln(v_3)]^{r_3-r_2-1}}{\Gamma(r_3-r_2)} \ldots \times \frac{[\ln (v_i)]^{r_i - r_{i-1}-1}}{\Gamma(r_i - r_{i-1})} (v_1 v_2 v_3 \ldots v_i)^{-kc-1}.$$

This prove the result.

Corollary 4.1 Suppose m and n are positive integers such that $1 \le m < n$ and $X_{U(m,k)}$ and $X_{U(n,k)}$ are mth and nth GURV arising from standard PD defined in (1.4). Then

$$E\left\{\frac{X_{U(n,k)}}{X_{U(m,k)}}\right\} = \frac{E\left(X_{U(n,k)}\right)}{E\left(X_{U(m,k)}\right)}. \tag{4.4}$$

Proof.

We have

$$E\left(X_{U(n,k)}\right) = E\left(\left\{\frac{X_{U(n,k)}}{X_{U(m,k)}}\right\} X_{U(m,k)}\right).$$

From the independence of $\frac{X_{U(n,k)}}{X_{U(m,k)}}$ and $X_{U(m,k)}$ (see, Theorem 4.1) we write

$$E\left(X_{U(n,k)}\right) = E\left\{\frac{X_{U(n,k)}}{X_{U(m,k)}}\right\} E\left(X_{U(m,k)}\right).$$

Thus we have $E\left\{\frac{X_{U(n,k)}}{X_{U(m,k)}}\right\} = \frac{E\left(X_{U(n,k)}\right)}{E\left(X_{U(m,k)}\right)}$. This is clearly a rare occasion experienced in statistics where expectation of a ratio of random variables is equal to ratio of the expectations of the individual random variables. More generally we get

$$E\left\{\frac{X_{U(n,k)}}{X_{U(m,k)}}\right\}^a = \frac{E\left(X_{U(n,k)}\right)^a}{E\left(X_{U(m,k)}\right)^a}$$

for any positive integer a.

Theorem 4.1 Suppose $X_{U(1,k)}, X_{U(2,k)}, \ldots, X_{U(n,k)}$ are the first n GURVs arising from the standard PD. Then the random variables $V_1 = X_{U(1,k)}, V_2 = \frac{X_{U(2,k)}}{X_{U(1,k)}}$, $V_3 = \frac{X_{U(3,k)}}{X_{U(2,k)}}, \ldots, V_{n-1} = \frac{X_{U(n-1,k)}}{X_{U(n-2,k)}}$ and $V_n = \frac{X_{U(n,k)}}{X_{U(n-1,k)}}$ are all statistically independent with the pdf of V_j's given by $h_j(v_j) = kcv_j^{-kc-1}, v_j > 1, j = 2,3, \ldots, n$.

Proof.

By putting $i = n$ and $r_1 = 1, r_2 = 2, \ldots, r_n = n$ in Theorem 4.1, we get the required result.

5. ESTIMATION OF THE PARAMETERS OF PARETO DISTRIBUTION

In this section we consider the situation in which only generalized (k) records for a given k alone are retrieved and made available on the variable which has the pdf given in (1.3). Now our interest is to estimate the parameters involved in (1.3) using the available GURVs.

5.1 Estimation of the Scale Parameter σ of Pareto Distribution when the Shape Parameter c is Known

Let $X_{U(1,k)}, X_{U(2,k)}, \ldots, X_{U(n,k)}$ be the n GURVs available from a PD as defined in (1.3) with known c. Then clearly $Y_{U(i,k)} = \frac{X_{U(i,k)}}{\sigma}, i = 1,2,\ldots,n$ are distributed as GURV arising from the standard Pareto distribution defined in (1.4). Assume that mean and variance of $Y_{U(i,k)}$ are finite (when $kc > 2$) for $1 \leq i \leq n$. Then from (2.3) and (2.5) we write $\alpha_n^{(k)} = E\left(X_{U(n,k)}\right) = \left(\frac{kc}{kc-1}\right)^n$ and $\sigma_{m,n}^{(k)} = Cov\left(Y_{U(m,k)}, Y_{U(n,k)}\right) = \frac{(kc)^n}{(kc-1)^{n-m}(kc-2)^m} - \left(\frac{kc}{kc-2}\right)^{n+m}$ for $1 \leq m \leq n$, where we define $Var\left(Y_{U(i,k)}\right) = \sigma_{i,i}^{(k)}, i = 1,2,\ldots,n.$

Let $\boldsymbol{X} = \left[X_{U(1,k)}, X_{U(2,k)}, \ldots, X_{U(n,k)}\right]^T$ and $\boldsymbol{Y} = \left[Y_{U(1,k)}, Y_{U(2,k)}, \ldots, Y_{U(n,k)}\right]^T$. Then the mean vector $E(\boldsymbol{Y})$ and the dispersion matrix $D(\boldsymbol{Y})$ of $\boldsymbol{Y}$ are given by $E(\boldsymbol{Y}) = \boldsymbol{\alpha}^{(k)} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ and $D(\boldsymbol{Y}) = \boldsymbol{\Sigma}^{(k)} = \left(\left(\boldsymbol{\sigma}_{i,j}^{(k)}\right)\right)$. Then the BLUE of σ is (see, Arnold *et al.*, 1998, p. 127)

$$\sigma^* = \frac{\boldsymbol{\alpha}^{(k)^T}\left[\boldsymbol{\Sigma}^{(k)}\right]^{-1}}{\boldsymbol{\alpha}^{(k)^T}[\boldsymbol{\Sigma}^{(k)}]^{-1}\boldsymbol{\alpha}^{(k)}}\boldsymbol{X} = \sum_{j=1}^{n} b_{j:n}X_{U(j,k)}. \tag{5.1}$$

Since the covariance terms involved in $\boldsymbol{\Sigma}^{(k)}$ can be written in the form

$$\sigma_{m,n}^{(k)} = Cov\left(Y_{U(m,k)}, Y_{U(n,k)}\right) = p_m q_n, \ for\ m < n, kc > 2,$$

where $p_m = \left[\left(\frac{kc-1}{kc-2}\right)^m - \left(\frac{kc}{kc-1}\right)^m\right]$ and $q_n = \left(\frac{kc}{kc-1}\right)^n$, the covariance matrix $\boldsymbol{\Sigma}^{(k)}$ can be inverted as given in Arnold *et al.* (1998, p. 128) analytically. Then the $(i,j)th$ element $e_{ij}^{(k)}$ of $\left[\boldsymbol{\Sigma}^{(k)}\right]^{-1}$ can be derived as

$$e_{ii}^{(k)} = \frac{\{[kc-1]^4 - [kc(kc-2)]^2\}(kc-2)^i}{(kc)^i}, \quad i = 1,2,3,\ldots,n-1$$

$$e_{nn}^{(k)} = \frac{(kc-1)^2(kc-2)^n}{(kc)^n}$$

$$e_{ij}^{(k)} = \frac{-(kc-1)(kc-2)^{i+1}}{(kc)^i}, \quad j = i+1, \qquad i = 1,2,\ldots,n-1$$

$$e_{ij}^{(k)} = 0, j > i+1.$$

Using $\left[\boldsymbol{\Sigma}^{(k)}\right]^{-1}$ obtained, in equation (5.1) and on simplification we obtain the closed form representation of σ^* as

$$\sigma^* = \left(\frac{kc-1}{kc}\right)X_{U(1,k)}. \tag{5.2}$$

The variance of the estimator σ^* is given by

$$Var(\sigma^*) = \frac{\sigma^2}{\boldsymbol{\alpha}^{(k)^T}[\boldsymbol{\Sigma}^{(k)}]^{-1}\boldsymbol{\alpha}^{(k)}}. \tag{5.3}$$

It is to be noted that the estimate σ^* given in (5.2) is also the unbiased estimator based on the MLE of σ.

5.2 Estimates of the Parameters c and σ When Both are Unknown

The joint density function of the first n GURVs is given by (Saran and Pandey, 2004)

$$f_{X_{U(1,k)},X_{U(2,k)},\dots,X_{U(n,k)}}(x_1,x_2,\dots,x_n) = k^n \prod_{i=1}^{n-1} \frac{f(x_i)}{1-F(x_i)} \left(1-F(x_n)\right)^{k-1} f(x_n), \qquad (5.4)$$

where $x_1 < x_2 < \cdots < x_n$. Using (5.4), the likelihood function in this case is given by

$$L = \left(\frac{kc}{\sigma}\right)^n \prod_{i=1}^{n-1} \left(\frac{x_i}{\sigma}\right)^{-1} \left(\frac{x_n}{\sigma}\right)^{-kc-1}, \qquad \sigma \le x_1 < x_2 < \cdots < x_n < \infty.$$

Hence

$$\log L = n\log k + n\log c - n\log\sigma - \sum_{i=1}^{n-1} \log\left(\frac{x_i}{\sigma}\right) - (kc+1)\log\left(\frac{x_n}{\sigma}\right).$$

The likelihood equations are then given by

$$\frac{\partial \log L}{\partial \sigma} = \frac{kc}{\sigma} = 0 \qquad (5.5)$$

$$\frac{\partial \log L}{\partial c} = \frac{n}{c} + k\log\sigma - k\log X_{U(n,k)} = 0. \qquad (5.6)$$

As we cannot get any direct solution for the parameter from (5.5), we consider (5.5) and with respect to the range of the GURVs, $X_{U(1,k)}$ is the closest value to σ and hence from the arguments put forward in David and Nagaraja (2003, p.173) we take $\hat{\sigma} = X_{U(1,k)}$. Using $\hat{\sigma}$ the estimate of c can be obtained as

$$\hat{c} = \frac{n}{k\log\left(\frac{X_{U(n,k)}}{\hat{\sigma}}\right)}. \qquad (5.7)$$

Further using Theorem 4.1 for $i = 2, r_1 = 1$ and $r_2 = n$ we have

$$E(\hat{c}) = \frac{(kc)^{n-1}}{\Gamma(n)} \int_1^\infty \frac{n}{k\log(v)} [\log(v)]^{n-1} v^{-kc-1} dv, \\ = \frac{n}{n-2} c.$$

Note that $\hat{c}$ is biased estimator for c. The unbiased estimator $\hat{\hat{c}}$ for c is given by

$$\hat{\hat{c}} = \frac{n-2}{n} \hat{c}$$

$$= \frac{n-2}{k \log\left(\frac{X_{U(n,k)}}{\hat{\sigma}}\right)}, \tag{5.8}$$

and it can be easily verified that for $n > 3$,

$$Var\left(\hat{c}\right) = \frac{c^2}{n-3}.$$

6. PREDICTION OF FUTURE GURV USING BLUP

One important problem on the quantitative study of a random variable is about its prediction for its future occurrence by using the information relating to the quantities measured on the variable during the immediate past period. In the available literature one can observe that considerable works have been carried out dealing with the prediction of future records. For more details on prediction of future records one may refer Gulati and Padgett (2003). In this section we deal with the construction of Best Linear Unbiased Predictor of a future GURV based on the values recorded earlier for the case of PD.

Suppose c is known and $X_{U(1,k)}, X_{U(2,k)}, \ldots, X_{U(n,k)}$ are the n available GURVs from PD with pdf given in (1.3). Then our interest here is on predicting the next GURV $X_{U(n+1,k)}$. The best linear unbiased predicted value of the next record can be obtained (see, Goldberger, 1968, Robinson, 1991 and Arnold *et al.*, 1998) as

$$X^*_{U(n+1,k)} = \alpha^{(k)}_{n+1}\sigma^* + w^T\left[\Sigma^{(k)}\right]^{-1}\left(X - \sigma^*\alpha^{(k)}\right), \tag{6.1}$$

where $\boldsymbol{X}$ is the vector of the initial n observed generalized upper (k) record values from PD as defined in (1.3), $\alpha^{(k)}$ and $\Sigma^{(k)}$ are the vector of means and dispersion matrix of the vector of n GURVs arising from standard PD, $\alpha^{(k)}_{n+1}$ is the expected value of $(n+1)th$ GURV arising from the standard PD and $\boldsymbol{w}^T = \left(\sigma^{(k)}_{1,n+1}, \sigma^{(k)}_{2,n+1}, \ldots, \sigma^{(k)}_{n,n+1}\right)$ is the vector of covariance of $X_{U(n+1,k)}$ with the n initial GURVs $X_{U(i,k)}$, $i = 1,2,\ldots,n$. The equation (6.1) can also be simplified and it is given by

$$X^*_{U(n+1,k)} = \left(\frac{kc}{kc-1}\right)X_{U(n,k)}, \tag{6.2}$$

and the variance of $X^*_{U(n+1,k)}$ is given by

$$Var\left(X^*_{U(n+1,k)}\right) = \left(\frac{kc}{kc-1}\right)^2\left[\left(\frac{kc}{kc-2}\right)^n - \left(\frac{kc}{kc-1}\right)^{2n}\right]\sigma^2. \tag{6.3}$$

7. NUMERICAL EXAMPLE

For illustration purpose we consider the annual wage data reported by Dyer (1981) (in multiplies of 100 U.S. dollars) of a random sample of 30 production-line workers in a large industrial firm. Dyer (1981) further established that PD shows a good fit for this data. The generalized (2) upper records obtained from these data set are given below

112,119,154,156,157

Clearly when both c and σ of PD are unknown the MLE of σ and c are given by $\hat{\sigma} = X_{U(1,k)}$ and $\hat{c} = \frac{n}{k\log\left(\frac{X_{U(n,k)}}{\hat{\sigma}}\right)}$ and the unbiased estimate $\hat{\hat{c}}$ of c based on $\hat{c}$ is

given by $\hat{\hat{c}} = \frac{n-2}{n}\hat{c}$. Now $\sigma_1 = \hat{\sigma}$ and $c_1 = \hat{\hat{c}}$ may be taken as the initial trial values of the estimators of σ and c respectively. Now using c_1 as the known value of c one can use σ^* given in (5.2) as the second stage estimator σ_2 of σ. Using σ_2 in (5.8) we then obtain the second stage estimator c_2 of c. We repeat this iteration repeatedly until the estimators of c and σ obtained in two successive iterations are not mutually different. For the given data the final estimate of c and σ are 8.702 and 105.564. Now using the value of c in the last iteration as the known value of c with respect the data we have used, we use the formula given in (6.2) to predict the 6th generalized (2) upper record value and is estimated as 166.571.

ACKNOWLEDGEMENTS

The authors express their gratefulness for the constructive comments of the learned referees which helped to improve considerably the revised version of the paper. The first author is much grateful to the University Grants Commission, Government of India for the financial support provided for this research work in the form of MANF-SRF. The second author expresses his gratefulness to Kerala State Council for Science, Technology & Environment for supporting him financially in the form of Emeritus Scientist Award.

REFERENCES

Abd-El-Hakim, N. S. and Sulthan, K. S. (2004): Maximum likelihood estimation from record- breaking data for the generalized Pareto distribution, *METRON - International Journal of Statistics,* **42,** 377-389.

Aczel, J. (1966): *Lectures on Functional Equations and Their Applications,* Academic Press, New York.

Ahsanullah, M. (1998): *Record Values: Theory and Applications,* University Press of America Inc., Lanham, Maryland, USA.

Ahsanulah, M. and Shakil, M. (2012): A note on the characterizations of Pareto distribution by upper record values, *Commun. Korean Math. Soc.,* **27,** 835-842.

Arnold, B. C. (1985): Pareto Distributions, In *Encyclopedia of Statistical Sciences,* New York: Wiley., 568-574.

Arnold, B. C. (2008): Pareto and Generalized Pareto Distributions, In *Modeling Income Distributions and Lorenz Curves,* Springer Science+Business Media, LLC, New York, 119-147.

Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998): *Records,* John Wiley and Sons, New York.

Balakrishnan, N. and Ahsanullah, M. (1994): Recurrence relations for single and product moments of record values from generalized Pareto distribution, *Comm. Statist.-Theory Methods,* **23,** 2841-2852.

Chandler, K. N. (1952): The distribution and frequency of record values, *J. Roy. Statist. Soc. Ser. B,* **14,** 220-228.

Chang, S. K. (2009): Characterizations of the Pareto distribution by record values, *J. Appl. Math.* & *Informatics,* **27,** 955-961.

Dallas, A. C. (1982): Some results on record values from the exponential and Weibull law, *Acta Math. Acad. Sci. Hungar*., **40,** 307-311.

David, H. A. and Nagaraja, H. N. (2003): *Order Statistics*, John Wiley and Sons, New York.

Dyer, D. (1981): Structural probability bounds for the strong Pareto law, *Canadian Journal of Statistics*, **9,** 71-77.

Dziubdziela, W. and Kopocinski, B. (1976): Limiting properties of the kth record values, *Appl. Math.,* **15,** 187-190,

Goldberger, S. (1962): Best linear unbiased prediction in the generalized linear regression model, *J. Amer. Statist. Assoc*., **57,** 369-375.

Gulati, S. and Padgett, W. J. (2003): *Parametric and Nonparametric Inference from Record-Breaking Data,* Springer-Verlag, New York.

Johnson, S. L., Kotz, S. and Balakrishnan, N. (1994): *Continuous Univariate Distributions,* John Wiley & Sons, Inc, New York.

Lee, M. Y and Chang, S. K. (2005): Characterizations based on the independence of the exponential and Pareto distributions by record values, *J. Appl. Math. & Computings,* **18,** 497-503.

Lee, M. Y and Lim, E. H. (2009): Characterizations of the Lomax, Exponential and Pareto distributions by conditional expectations of record values, *J. Chungcheong. Math. Soc.,* **22,** 149-153.

Lee, M. Y and Lim, E. H. (2011): On characterizations of the Pareto distribution by the independent property of upper record values, *J. Chungcheong. Math. Soc.,* **24,** 85-89.

Minimol, S. and Thomas, P. Y (2013): On some properties of Makeham distribution using generalized record values and its characterizations, *Braz. J. Probab. Stat.,* **27,** 487-501.

Minimol, S. and Thomas, P. Y (2014): On characterization of Gompertz distribution by generalized record values, *J. Stat. Theory Appl.,* **13,** 38-45

Nevzorov, V. (2001): *Records: Mathematical Theory*. Translation of Mathematical Monographs, vol. 194. American Mathematical Society, Providence, RI, USA.

Pareto, V. (1897): *Cours d'Economie Politique*. Rouge, Lausanne.

Paul, J. and Thomas, P. Y (2015): On generalized upper (k) record values from Weibull distribution, *Statistica,* **75,** 313-330.

Raqab, M. Z. and Awad, A. M. (2000): Characterizations of Pareto and related distributions, *Metrika,* **52,** 63-67.

Robinson, G. K. (1991): That BLUP is a good thing: The estimation of random effects, *Stat. Sci.,* **6,** 15-51.

Saran, J. and Pandey, A. (2004): Estimation of parameters of a power function distribution and its characterization by k-th record values, *Statistica,* **64,** 523-536.

Sulthan, K. S. and Moshref, M. E. (2000): Record values from generalized Pareto distribution and associated inference, *Metrika,* **51,** 105-116.

Wu, J. W. and Lee, W. C. (2001): On the characterization of generalized extreme value, power function, generalized Pareto and classical Pareto distributions by conditional expectation of record values, *Statist. Papers,* **42,** 225-242.