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Characterization Of Exponential Distribution By Spacing Of Records

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ABSTRACT

In this paper, exponential distribution is characterized by distributional properties of record statistics using normalized spacing. The distribution of difference of two nonadjacent records is being considered to characterize the exponential distributions. The necessary and sufficient condition has been proved.

1. INTRODUCTION

Extreme values frequently occur in lifetime data analysis. The choice of taking record values can be justified as in many real life situations, where the data are available in the form of extremes - i.e. only those data are recorded which are higher/lower than the previous highest/lowest value. Record values are widely used in extreme weather conditions, sports, economics, insurance reliability and in many other real life perspectives. For the application of record values in various discipline, one may refer to Balakrishnan *et al.* (1993), Arnold *et al.* (1998), Balakrishnan and Chan (1998), Katz *et al.* (2002), Raqab (2002), Ahsanullah (2004), Ahmadi *et al.* (2009) and Wergen (2014). Record values were first defined by Chandler (1952) as a model of successive extremes in a sequence of identical and independent random variables as follows:

Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed (*iid*) continuous random variables (*rv*) with the cumulative distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$ over the support (α, β). Define the upper record times by $U(1) = 1$ and for $r > 1$,

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$U(r) = \min\{k > U(r-1) : X_k > X_{U(r-1)}\}$. The upper record value sequence is then defined by $X_{U(1)}, X_{U(2)}, \ldots$.

The pdf of r^{th} upper record $X_{U(r)}$ is given (Ahsanullah; 2004) as

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}[-\ln \bar{F}(x)]^{r-1} f(x), \; -\infty < x < \infty \tag{1.1}$$

where $\bar{F}(x) = 1 - F(x)$ denotes the survival function.

The joint pdf of $X_{U(r)}$ and $X_{U(s)}$, $r < s$, is given (Ahsanullah ; 2004) as

$$f_{X_{U(r)}X_{U(s)}}(x, y) = \frac{[-\ln \bar{F}(x)]^{r-1}}{(r-1)!} \frac{[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1}}{(s-r-1)!} \frac{f(x)}{\bar{F}(x)} f(y),$$
$$-\infty < x < y < \infty \tag{1.2}$$

For comprehensive accounts of the theory and applications of record values in book form, one may refer to Arnold *et al.* (1998), Ahsanullah (2004) and Ahsanullah and Raqab (2006).

Let $X_1, X_2, \ldots X_n$ be independently and identically distributed (iid) random variables from exponential distribution with shape parameter λ, i.e $F(x) = 1 - e^{-\frac{x}{\lambda}}, x > 0, \lambda > 0$ and $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ are the corresponding orders statistics. The normalized spacing of order statistics (when the random variables are *iid* and from exponential distribution) was introduced by Sukhatme (1937). Sukhatme (1937) considered the transformation $D_{1:n} = nX_{1:n}$ and $D_{r:n} = (n - r + 1)(X_{r:n} - X_{r-1:n})$, $2 \le r \le n$. He proved that $D_{r:n}$, $1 \le r \le n$ are *iid* from exponential distribution with shape parameter λ. Utilizing the above transformation, it can be shown that $X_{s:n} - X_{r:n}$ and $X_{s-r:n-r}$ are identically distributed for all $1 \le r < s \le n$.

Based on the order statistics, the characterization of exponential distribution through normalized spacing was considered by several authors. To avail relevant literature on this topic, one may refer to Ahsanullah (1981, a), Iwińska (1986), Gather (1988) and Gajeck and Gather (1989) and the references cited therein. Characterization results involving spacing of record statistics were also considered by several authors. Tata (1969) has shown that the independence of $X_{U(2)} - X_{U(1)}$ and $X_{U(1)}$ is a characteristic property of exponential distribution. The related results of characterization of exponential distribution for record values through spacing was shown by Ahsanullah (1981, b) and Iwińska (1986).

Ahsanullah (1981 b) showed that if $E(X_{U(n} - X_{U(m)}) = E(X_{U(n-m)}),\ n > m,$ then the random variable X is exponentially distributed.
In this paper, we have characterized the exponential distribution through normalized spacing based on record statistics. Similar characterization result was obtained by Ahsanullah (1987) but the approach used in this paper is entirely different. The paper is divided into two sections. In section 2, we have characterized the exponential distribution through the spacing of record statistics.

2. CHARACTERIZATION OF EXPONENTIAL DISTRIBUTION

In this section we have characterized the exponential distribution based on record statistics. Before the proof of the main result, the following two lemmas are given, which are used in the proof of the theorem. Proofs of the Lemma's are straightforward and hence omitted.

Lemma 2.1: Let $X_{U(r-1)}$ and $X_{U(r)}$ be the two adjacent upper records, then

$$\bar{F}_{X_{U(r)}}(x) - \bar{F}_{X_{U(r-1)}}(x) = \frac{[-\ln \bar{F}(x)]^{r-1}}{(r-1)!}.$$

Lemma 2.2: The function $h(x) = [-\ln \bar{F}(x)]^{r-1}\bar{F}(x)$ is strictly increasing with maxima occurring at point $x = F^{-1}[1 - e^{-(r-1)}]$ and then strictly decreasing.
Now we shall prove the main result of this paper.
Theorem 2.1: Let X be a non negative continuous random variable with strictly increasing (with respect to Lebesgue measure) df $F(x)$ and pdf $f(x)$ over the support $(0, \infty)$, then for $1 \le r < s-1 \le n-1$,

$$X_{U(l)} - X_{U(r)} \underline{\underline{d}} X_{U(l-r)}, \qquad l = s-1, s$$

if and only if

$$F(x) = 1 - e^{-\frac{x}{\lambda}},$$

Where the symbol $\underline{\underline{d}}$ denotes the equality in distribution.
Proof: For the proof of the necessary part, using the normalized spacing of record values from exponential distribution i.e. $Y_1 = X_{U(1)}, Y_i = X_{U(i)} - X_{U(i-1)}$, $i = 2,3,\dots$, it can be seen that $Y_1, Y_2, \dots$ are iid random variables with exponential distribution with the scale parameter λ. Thus $X_{U(l)} - X_{U(r)} = \sum_{i=r+1}^{l} Y_i,\ l = s-1, s.$ Then $X_{U(l)} - X_{U(r)}$ is gamma distributed with shape parameter as $l - r$ and scale parameter λ. Further when random variable X follows exponential distribution with scale parameter λ, then

distribution of $X_{U(l-r)}$ is also gamma with shape parameter as $l-r$ and scale parameter λ. This implies that $X_{U(l)} - X_{U(r)} \underset{=}{d} X_{U(l-r)}$, $l = s-1, s$.
Now to prove the sufficient part, we have for any positive and finite u,

$$P[(X_{U(l)} - X_{U(r)}) \geq u] = \int_0^\infty \int_{x+u}^\infty \frac{[-\ln \bar{F}(x)]^{r-1}}{(r-1)!} \frac{[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1}}{(s-r-1)!} \frac{f(x)}{\bar{F}(x)} f(y)\, dx\, dy \tag{2.1}$$

Let $A(x,y) = [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1} f(y)$.

Integrating $A(x,y)$ w.r. to y over $(x+u, \infty)$

$$\int_{x+u}^\infty A(x,y)\, dy = \int_{x+u}^\infty [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1} f(y)\, dy$$

$$= (s-r-1) \int_{x+u}^\infty [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-2} f(y)\, dy + [\ln \bar{F}(x) - \ln \bar{F}(x+u)]^{s-r-1} \bar{F}(x+u) \tag{2.2}$$

Substituting the value of $\int_{x+u}^\infty A(x,y)\, dy$ obtained in (2.2) in (2.1), we have

$$P[(X_{U(s)} - X_{U(r)}) \geq u] =$$

$$\frac{1}{(r-1)!(s-r-1)!} \int_0^\infty \int_{x+u}^\infty [-\ln \bar{F}(x)]^{r-1} [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-2} \frac{f(x)}{\bar{F}(x)} f(y)\, dx\, dy$$

$$+ \frac{1}{(r-1)!(s-r-1)!} \int_0^\infty [-\ln \bar{F}(x)]^{r-1} [-\ln \bar{F}(x+u) + \ln \bar{F}(x)]^{s-r-1} \frac{\bar{F}(x+u)}{\bar{F}(x)} f(x)\, dx$$

$$P[(X_{U(s)} - X_{U(r)}) \geq u] - P[(X_{U(s-1)} - X_{U(r)}) \geq u]$$

$$= \frac{1}{(r-1)!(s-r-1)!} \int_0^\infty [-\ln \bar{F}(x)]^{r-1} [-\ln \bar{F}(x+u) + \ln \bar{F}(x)]^{s-r-1} \frac{\bar{F}(x+u)}{\bar{F}(x)} f(x)\, dx \tag{2.3}$$

Now in view of lemma 2.1 and utilizing the fact that $f_{X_{U(r)}}(x)$ is a *pdf* , we have

$$P[(X_{U(s-r)}) \geq u] - P[(X_{U(s-r-1)}) \geq u] = \frac{\bar{F}(u)[-\ln \bar{F}(u)^{s-r-1}}{(r-1)!(s-r-1)!} \int_0^\infty [-\ln \bar{F}(x)]^{r-1} f(x)\, dx. \tag{2.4}$$

Equation (2.3) and (2.4) will be equal if and only if

$$=\frac{1}{(r-1)!(s-r-1)!}\times$$

$$\times\int_0^\infty\left[[-\ln\bar{F}(x+u)+\ln\bar{F}(x)]^{s-r-1}\frac{\bar{F}(x+u)}{\bar{F}(x)}-[-\ln\bar{F}(u)]^{s-r-1}\bar{F}(u)\right][-\ln\bar{F}(x)]^{r-1}f(x)dx=0.$$

Since $[-\ln\bar{F}(x)]^{r-1}$ and $f(x)$ both are positive, using the generalization of Muntz-Swartz theorem (see Hwang Lin; 1984), the above integral will be zero if and only if

$$[-\ln\bar{F}(x+u)+\ln\bar{F}(x)]^{s-r-1}\frac{\bar{F}(x+u)}{\bar{F}(x)}=[-\ln\bar{F}(u)]^{s-r-1}\bar{F}(u).$$

Now in view of lemma 2.2, the function $h(x)=[-\ln\bar{F}(x)]^{s-r-1}\bar{F}(x)$ is strictly increasing and has maxima at point $x=F^{-1}[1-e^{-(r-1)}]$ (see Basak and Basak, 2002; Cramer *et.al,* 2004). Thus the L.H.S and R.H.S are equal only at maxima, which is satisfied only with the condition i.e. $\bar{F}(x+u)=\bar{F}(x)\bar{F}(u)$ which is nothing but a memory-less property. The only continuous distribution with the boundary condition $F(0)=0$ and $F(\infty)=1$, which satisfies $\bar{F}(x+u)=\bar{F}(x)\bar{F}(u)$ is exponential distribution *i.e.* $F(x)=1-e^{-\frac{x}{\lambda}}$. This implies that the random variable X follows exponential distribution with scale parameter λ and hence the theorem.

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REFERENCES

Ahmadi, J., Jozani, M. J., Marchand, E. and Parsian, A. (2009): Prediction of k-records from a general class of distributions under balanced type loss functions. *Metrika,* **70**, 19-33.

Ahsanullah, M. (1981 a): On characterizations of the exponential distribution by spacings. *Statist. Hefte*., **22**, 121-127.

Ahsanullah, M. (1981 b): Record values of exponentially distributed random variables, *ibid*., **22**, 316-320.

Ahsanullah, M. (1987): Two characterizations of the exponential distributions. *Comm. Statist. Theory Methods,* **16(2)**, 375-381.

Ahsanullah, M. (2004): *Record Values-Theory and Applications*. University Press of America, New York.

Ahsanullah, M. and Raqab, M. Z. (2006): *Bounds and Characterizations of Record Statistics*. Nova Science Publishers, Hauppauge, New York.

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998): *Records*. Wiley, NY.

Balakrishnan, N., Chan, P. S. and Ahsanullah, M. (1993): Recurrence relations for moments of record values from generalized extreme value distribution. *Comm. Statist. Theory Methods,* **22**, 1471-1482.

Balakrishnan, N. and Chan, P. S. (1998): On the normal record values and associated inference. *Statist. Probab. Lett.*, **39**, 73-80.

Basak,P. and Basak, I. (2002): Unimodality of the distribution of record statistics. *Statist. Probab. Lett.*, **56** , 395–398.

Chandler, K. N. (1952): The distribution and frequency of record values. *J. R. Statist. Soc.,* **B14**, 220-228.

Cramer, E., Kamps, U. and Rychlik, T. (2004): Unimodality of uniform generalized order statistics with applications to mean bounds. *Ann. Inst. Statist. Math*., **56**, 183-192.

Gajec, L. and Gather, U. (1989): Characterizations of the exponential distribution by failure rate and moment properties of order statistics, in: *J. Husler and R. D.Reiss (eds.), Extreme Value Theory, Springer, Berlin*, 114-124.

Gather, U. (1988): On a characterization of the exponential distribution by properties of order statistics. *Statist. Probab. Lett*., **7**, 93-96.

Iwińska, M. (1986): On the characterizations of the exponential distribution by order statistics and record values. *Fasc. Math*., **16**, 101-107.

Katz, R. W., Parlange, M. B. and Naveau, P. (2002): Statistics of extremes in hydrology. *Adv. Water Resour.*, **25**, 1287-1304.

Hwang, J. S. and Lin, G. D. (1984): On a generalized moments problem II. *Proc. Amer. Math.Soc*. **91**, 577-580.

Raqab, M. Z. (2002): Inferences for generalized exponential distribution based on record statistics. *J. Statist. Plann. Inference,* **104**, 339-350.

Sukhatme, P. V. (1937): Tests of significance for samples of the χ^2 population with two degrees of freedom, *Ann. Eugenics,* **8**, 52-56.

Tata, M. N. (1969) : On outstanding values in a sequence of random variables. *Zeit. Wahrscheinlichkeitstheorie Verw. geb*., **12**, 9-20.

Wergen, G. (2014): Modeling record-breaking stock prices. *Physica A,* **396**, 114-133.