Aligarh Journal of Statistics
Vol. 30 (2010), 23-38

INFERENCES ON THE SYSTEMS WITH GUARANTEED STRENGTH SUBJECTED TO SOME AT LEAST STRESS

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ABSTRACT

With the increasing awareness and desire of consumers to have the best product in terms of product features like reliability, hazard rate etc., large number of equipments/systems in the market are, nowadays, facing hard competition to attract the attention of the targeted clients. In such situations, it is in the interest of the producers to attach some guarantee to the performance of the equipments/systems when these are tested or subjected to some stresses for their abilities to perform the intended functions. Thus, in the present paper, we, at first, consider the concept of re-modeling the stress-strength reliability model, $\Pr[X > Y]$, in the light of guaranteed strength and the at least stresses that are faced. Secondly, the re-modeled stress-strength reliability model has been analyzed both in the classical and Bayesian set-ups.

1. INTRODUCTION

The studies like Bhattacharya and Johnson (1974), Chao (1982), Harris and Church (1970), Kapur and Lamberson (1977), Porat *et al.* (1994) and Sathe and Dixit (2001) have reported a large literature on the classical estimation of stress-strength reliability, $\Pr[X > Y]$. This model is concerned with the reliability of a component's strength, X, subjected to a stress, Y. Assuming the random character of the parameters of the basic distributions of X and Y, the study in Draper and Guttman (1978) provided the Bayesian analysis of the reliability in multi-component stress-strength model. The study by Owen *et al.* (1964) developed a non-parametric approach for defining the confidence limits and confidence bounds for $\Pr[X > Y]$.

In practice, we come across situations where the disposal of the systems, equipments or establishments in the market has to face stiff competition in terms of their quality specifications. Accordingly, the designer's objective is to attach high probability to the event that the system performs its intended task satisfactorily with a guaranteed strength when subjected to some at least stress. For meeting the stated designer's objective, the present study initially deals with the remodeling of the strength-stress reliability model $\Pr[X > Y]$. This re-modeled reliability is then analyzed in both classical and Bayesian frameworks.

In the Classical setup, we obtain maximum likelihood estimate (MLE), uniformly minimum variance unbiased estimate $(UMVUE)$ along with their simulated sample mean square error (MSE) of re-modeled system reliability. On the other hand, using the subjective priors for the random parameters, Bayes estimates of re-modeled stress-strength reliability model under squared error loss function $(SELF)$ along with their posterior variances are obtained and analyzed. It is important to mention that Bayes estimates of the system reliability are calculated using numerical integration and Lindley approximation approaches.

For re-modeling the system reliability, the strength and stress variables are re-defined as:

$$X_{\mu_x} = \begin{cases} X & , \quad X > \mu_x \\ 0 & , \quad otherwise \end{cases}$$

and

$$Y_{\mu_y} = \begin{cases} Y & , \quad Y > \mu_y \\ 0 & , \quad otherwise \end{cases}$$

i.e. the strength $'X'$ and stress $'Y'$ variables are truncated in the ranges $[0, \mu_x]$ and $[0, \mu_y]$ respectively. Here, $'\mu_x'$ is the guaranteed strength and $'\mu_y'$ is the at-least stress encountered by the system during its operation. Here, both μ_x and μ_y are assumed to be known.

Hence, the re-modeled form, say R_{μ_x,μ_y}, of the system reliability, $\Pr[X > Y]$ is

$$\begin{aligned} R_{\mu_x,\mu_y} &= \Pr[X > Y \mid \mathrm{X} > \mu_x, \mathrm{Y} > \mu_y] \\ &= \Pr[\mathrm{X}_{\mu_x} > \mathrm{Y}_{\mu_y}]. \end{aligned} \tag{1.1}$$

Here, it is assumed that $'X'$ and stress $'Y'$ are stochastically independently distributed.

2. STATISTICAL BACKGROUND

For establishing the theoretical developments, it is assumed that:

i) The random variable (rv) $'X'$ follows Weibull distribution with probability density function (pdf)

$$f_X(x;\theta,c)=\frac{c\,x^{c-1}}{\theta}\exp\left\{-\left(\frac{x^c}{\theta}\right)\right\};\quad x>0\ ,(\theta,c)>0\,. \tag{2.1}$$

Here,

$$E(X)=\theta^{\frac{1}{c}}\,\Gamma\left(\frac{1}{c}+1\right)$$

$$V(X)=\theta^{\frac{2}{c}}\left[\Gamma\left(\frac{2}{c}+1\right)-\left\{\Gamma\left(\frac{1}{c}+1\right)\right\}^2\right].$$

ii) The rv $'Y'$ also follows Weibull distribution with pdf

$$f_Y(y;\lambda,a)=\frac{a\,y^{a-1}}{\lambda}\exp\left\{-\left(\frac{y^a}{\lambda}\right)\right\};\quad y>0\ ,\ (\lambda,a)>0\,. \tag{2.2}$$

Here,

$$E(Y)=\lambda^{1/a}\,\Gamma\left(\frac{1}{a}+1\right)$$

$$V(Y)=\lambda^{2/a}\left[\Gamma\left(\frac{2}{a}+1\right)-\left\{\Gamma\left(\frac{1}{a}+1\right)\right\}^2\right].$$

iii) On using i) above, the pdf of left truncated variable $'X'$, say $h(x)$, becomes

$$h(x)=\begin{cases}\dfrac{f_X(x;\theta,\ c)}{\int_T f_X(x;\theta,c)\,dx}\ ; & x\in T\quad,\quad T\equiv\{T: x>\mu_x\}\\ 0\ ; & otherwise\end{cases}$$

$$=\begin{cases}\dfrac{cx^{c-1}}{\theta}\exp\left\{-\left(\dfrac{x^c-\mu_x{}^c}{\theta}\right)\right\} & ;\quad x>\mu_x\,.\\ 0\ ; & otherwise\end{cases} \tag{2.3}$$

Similarly, in view of (ii) above, the pdf of the left truncated variable $'Y'$, say $h(y)$, becomes

$$h(y)=\begin{cases}\dfrac{a\,y^{a-1}}{\lambda}\exp\left\{-\left(\dfrac{y^{a}-\mu_y{}^{a}}{\lambda}\right)\right\}; & y>\mu_y\\ 0\quad; & otherwise\end{cases} \tag{2.4}$$

For Bayesian setup, it is further assumed that:

v) θ is considered as a rv following inverted gamma prior with pdf

$$h_1(\theta)=\frac{\alpha^{\beta}\exp\left\{-\left(\dfrac{\alpha}{\beta}\right)\right\}}{\theta^{\beta+1}\Gamma\beta};\qquad \theta>0,\ \ (\alpha,\beta)>0. \tag{2.5}$$

With

$$E(\theta)=\frac{\alpha}{\beta-1},\ \text{for }\ \beta>1;\qquad V(\theta)=\frac{\alpha^2}{(\beta-1)^2(\beta-2)},\ \text{for }\ \beta>2.$$

Here, the parameter c in (2.3) is assumed to be known.

vi) λ is also a rv following inverted gamma prior with pdf

$$h_2(\lambda)=\frac{\xi^{\tau}\exp\left\{-\left(\dfrac{\xi}{\tau}\right)\right\}}{\lambda^{\tau+1}\Gamma\tau};\qquad \lambda>0,\ \ (\xi,\tau)>0. \tag{2.6}$$

With

$$E(\lambda)=\frac{\xi}{\tau-1},\ \text{provided}\ \ \tau>1,$$

$$V(\lambda)=\frac{\xi^2}{(\tau-1)^2\,(\tau-2)},\ \text{provided}\ \ \tau>2.$$

Here, the parameter a in (2.4) is also assumed to be known.

vii) For the sample of size n say $\underline{X}=(X_1,X_2,\ldots,X_n)$ from (2.3), the joint density function of $\underline{X}$ is given by

$$L_1(\underline{x}\,|\,\theta,c)=L_1=\left(\frac{c}{\theta}\right)^{n}\prod_{i=1}^{n}x_i^{c-1}\exp\left\{-\sum_{i=1}^{n}\left(\frac{x_i^{c}-\mu_x^{c}}{\theta}\right)\right\} \tag{2.7}$$

Similarly, the joint density function of the sample of size m from (2.4), say $\underline{Y}=(Y_1,Y_2,\ldots,Y_n)$ is given by

$$L_2(\underline{y}\,|\,\lambda,a)=L_2=\left(\frac{a}{\lambda}\right)^m \prod_{j=1}^{m} y_j^{a-1}\exp\left\{-\sum_{j=1}^{m}\left(\frac{y_j^a-\mu_y^a}{\lambda}\right)\right\}. \tag{2.8}$$

3. RE-MODELED RELIABILITY IN PARAMETRIC TERMS

On using (2.3) and (2.4) in (1.1), the parametric form of the re-modeled reliability can be written as

$$R_{\mu_x,\mu_y}=\int_{\mu_x}^{\infty}\left[\int_{\mu_y}^{x}\frac{ay^{a-1}}{\lambda}\exp\left\{-\left(\frac{y^a-\mu_y^a}{\lambda}\right)\right\}dy\right]\frac{cx^{c-1}}{\theta}$$

$$\times\exp\left\{-\left(\frac{x^c-\mu_x^c}{\theta}\right)\right\}dx$$

$$=1-\int_{\mu_x}^{\infty}\frac{cx^{c-1}}{\theta}\exp\left[-\left\{\left(\frac{x^c-\mu_x^c}{\theta}\right)+\left(\frac{x^a-\mu_y^a}{\lambda}\right)\right\}\right]dx\,.$$

On using the transformation

$$g=\frac{x^c-\mu_x^c}{\theta}\quad\Rightarrow\quad 0<g<\infty\,.$$

The re-modeled system reliability reduces to

$$R_{\mu_x,\mu_y}=1-\int_0^{\infty}\exp\left[-\left\{g+\left(\frac{(\theta g+\mu_x^c)^{a/c}-\mu_y^a}{\lambda}\right)\right\}\right]dg\,. \tag{3.1}$$

4. ESTIMATION OF THE RE-MODELED RELIABILTY *MLE* OF R_{μ_x,μ_y}

The logarithm of the likelihood function given in (2.7) is

$$\log L_1(\underline{x}\,|\,\theta,c)=n\log c-n\log\theta+(c-1)\sum_{i=1}^{n}\log x_i-\frac{1}{\theta}\left(\sum_{i=1}^{n}x_i^c-n\mu_x^c\right)$$

To obtain *MLE s* $(\hat{c},\hat{\theta})$ of (c,θ), we consider

$$\frac{\partial\log L_1(\underline{x}\,|\,\theta,c)}{\partial\theta}=\frac{-n}{\theta}+\frac{1}{\theta^2}\left(\sum_{i=1}^{n}x_i^c-n\mu_x^c\right)=0$$

and

$$\frac{\partial \log L_1(\underline{x}\,|\,\theta, c)}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \log x_i - \frac{1}{\theta}\left(\sum_{i=1}^{n} x_i^c \log x_i - n\mu_x^c \log \mu_x\right) = 0$$

Thus, the $MLE\,s$ $\hat{c}$ and $\hat{\theta}$ are solutions of the equations

$$\frac{\left(\sum_{i=1}^{n} x_i^{\hat{c}} \log x_i - n\mu_x^{\hat{c}} \log \mu_x\right)}{\left(\sum_{i=1}^{n} x_i^{\hat{c}} - n\mu_x^{\hat{c}}\right)} - \frac{1}{\hat{c}} = \frac{1}{n}\sum_{i=1}^{n} \log x_i \tag{4.1}$$

and

$$\hat{\theta} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{\hat{c}} - n\mu_x^{\hat{c}}\right) \tag{4.2}$$

(4.1) may be solved for $\hat{c}$ by Newton Raphson method or any other suitable iterative method and accordingly, $\hat{\theta}$ is obtained from (4.2).

For known value of c, $\hat{\theta}$ will be

$$\hat{\theta} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{c} - n\mu_x^{c}\right).$$

Similarly, following as above, the respective $MLE\,s$ of a and λ i.e. $\hat{a}$ and $\hat{\lambda}$ are the solutions of

$$\frac{\left(\sum_{j=1}^{m} y_j^{\hat{a}} \log y_j - m\mu_y^{\hat{a}} \log \mu_y\right)}{\left(\sum_{j=1}^{m} y_j^{\hat{a}} - m\mu_y^{\hat{a}}\right)} - \frac{1}{\hat{a}} = \frac{1}{m}\sum_{j=1}^{m} \log y_j \tag{4.3}$$

and

$$\hat{\lambda} = \frac{1}{m}\left(\sum_{j=1}^{m} y_j^{\hat{a}} - m\mu_y^{\hat{a}}\right) \tag{4.4}$$

(4.3) may be solved for $\hat{a}$ by Newton Raphson method or other suitable iterative method and this value is substituted in (4.4) to obtain $\hat{\lambda}$.

For known value of a, $\hat{\lambda}$ becomes

$$\hat{\lambda} = \frac{1}{m}\left(\sum_{j=1}^{m} y_j^a - m\mu_y^a\right).$$

Finally, on using the invariance property of the *MLEs*, the *MLE* of the re-modeled system reliability in (3.1) becomes

$$\hat{R}_{\mu_x,\mu_y} = 1 - \int_0^\infty \exp\left[-\left\{g + \left(\frac{(\hat{\theta}g + \mu_x^{\hat{c}})^{\hat{a}/\hat{c}} - \mu_y^{\hat{a}}}{\hat{\lambda}}\right)\right\}\right] dg \tag{4.5}$$

For known values of a and c, the above expression in (4.5) reduces to

$$\hat{R}_{\mu_x,\mu_y} = 1 - \int_0^\infty \exp\left[-\left\{g + \left(\frac{(\hat{\theta}g + \mu_x^{c})^{a/c} - \mu_y^{a}}{\hat{\lambda}}\right)\right\}\right] dg \tag{4.6}$$

The expression for $\hat{R}_{\mu_x,\mu_y}$ given in equations (4.5) and (4.6) cannot be reduced in closed forms.

UMVUE OF R_{μ_x,μ_y}

Lemma 4.1: For equal shape parameters, $a = c$, the *UMVUE*, $\tilde{R}_{\mu_x,\mu_y}$, of the re-modeled reliability R_{μ_x,μ_y} is

$$\tilde{R}_{\mu_x,\,\mu_y} = \begin{cases} (m-1)(n-1)\displaystyle\sum_{i=0}^{m-2}\sum_{j=0}^{n-2}\sum_{k=0}^{i+1}(-1)^{i+j}\binom{m-2}{i}\binom{n-2}{j}\binom{i+1}{k} \\ \times \dfrac{(\mu_x^c - \mu_y^c)^{i+1-k}(v - n\mu_x^c)^k}{(i+1)(u - m\mu_y^c)^{i+1}(j+k+1)} \quad ; \quad v > n\mu_x^c, \quad u > m\mu_y^c \\ 0 \quad ; \quad otherwise \end{cases}$$

Proof: For $a = c$, the expression of the re-modeled reliability in (3.1) reduces to

$$R_{\mu_x,\mu_y} = 1 - \left(\frac{\lambda}{\lambda + \theta}\right)\exp\left\{-\left(\frac{\mu_x^c - \mu_y^c}{\lambda}\right)\right\}. \tag{4.7}$$

Now, let us define a statistic

$$k(x,y)=\begin{cases}1 & ; \quad if\ X_1^c > Y_1^c \\ 0 & ; \quad otherwise\end{cases}$$

The unbiased estimate of the re-modeled reliability expressed in (4.7) is $k(x,y)$.

Further, $\sum_{i=1}^{n} x_i^c$ and $\sum_{j=1}^{m} y_j^c$ are the respective complete sufficient statistics for θ and λ when c, μ_x and μ_y are known.

Hence, by Lehmann- Scheffe theorem, the *UMVUE* of the re-modeled reliability in (4.7) comes out to be

$$E\left[k(x,y)\,|\sum_{i=1}^{n} x_i^c, \sum_{j=1}^{m} y_j^c\right] = \Pr\left[(\mathrm{X}_1^c > \mathrm{Y}_1^c)\,|\sum_{i=1}^{n} x_i^c, \sum_{j=1}^{m} y_j^c\right]$$

$$= \Pr\left[\left(\mathrm{X}_1^c\,|\sum_{i=1}^{n} x_i^c\right) > \left(\mathrm{Y}_1^c\,|\sum_{j=1}^{m} y_j^c\right)\right]$$

$$= \Pr[(W=w|V=v) > (Z=z\,|U=u)], \tag{4.8}$$

where

$$W = X_1^c, \quad V=\sum_{i=1}^{n} X_i^c, \quad Z = Y_1^c \quad \text{and} \quad U=\sum_{j=1}^{m} Y_j^c\,.$$

In order to obtain the expression in (4.8), we have to derive the conditional distributions of $(W=w|V=v)$ and $(Z=z\,|U=u)$.

Now, on using (2.3), the respective distributions of W and V can be easily obtained

$$f_W(w)=\frac{1}{\theta}\exp\left\{-\left(\frac{w-\mu_x^c}{\theta}\right)\right\}; \quad w>\mu_x^c$$

and

$$f_V(v)=\frac{(v-n\mu_x^c)^{n-1}}{\theta^n\Gamma n}\exp\left\{-\left(\frac{v-n\mu_x^c}{\theta}\right)\right\}; \quad v>n\mu_x^c\,.$$

Further, $V-W=\sum_{i=2}^{n} X_i^c$ and has the pdf

$$f_{V-W}(v-w)=\frac{\{(v-w)-(n-1)\,\mu_x^c\}^{n-2}}{\theta^{n-1}\,\Gamma(n-1)}\exp\left[-\left\{\frac{(v-w)-(n-1)\,\mu_x^c}{\theta}\right\}\right];$$

$$(v-w)>(n-1)\,\mu_x^c\,.$$

Following the study in Sinha (1986), the *rvs* W and $(V-W)$ are independent. Hence, the conditional distribution of $(W\,|\,V=v)$) is

$$f\,(W=w\,|\,V=v)=\frac{f\,(w,v)}{f\,(v)}=\frac{f\,(w)\,f\,(v-w)}{f\,(v)}$$

$$=\left(\frac{n-1}{v-n\mu_x^c}\right)\left\{1-\left(\frac{w-\mu_x^c}{v-n\mu_x^c}\right)\right\}^{n-2};$$

$$v>n\mu_x^c\,,\qquad \mu_x^c<w<v-(n-1)\,\mu_x^c \tag{4.9}$$

Similarly, in view of (2.4), the conditional distribution of $(Z\,|\,U=u)$ is

$$f\,(Z=z\,|\,U=u)=\frac{f\,(z,u)}{f\,(u)}=\frac{f\,(z)\,f\,(z-u)}{f\,(u)}$$

$$=\left(\frac{m-1}{u-m\mu_y^c}\right)\left\{1-\left(\frac{z-\mu_y^c}{u-m\mu_y^c}\right)\right\}^{m-2};$$

$$u>m\mu_y^c,\qquad \mu_y^c<z<u-(m-1)\mu_y^c \tag{4.10}$$

Now, using (4.9) and (4.10) in (4.8), the *UMVUE* of R_{μ_x,μ_y} in (4.7), say $\tilde{R}_{\mu_x,\mu_y}$, will be

$$\tilde{R}_{\mu_x,\mu_y}=\begin{cases}\displaystyle\int_{\mu_x^c}^{\{v-(n-1)\mu_x^c\}}\left[\int_{\mu_y^c}^{w}\left(\frac{m-1}{u-m\,\mu_y^c}\right)\left\{1-\frac{z-\mu_y^c}{u-m\,\mu_y^c}\right\}^{m-2}dz\right]\left(\frac{n-1}{v-n\mu_x^c}\right) & \\ \qquad\times\left\{1-\left(\frac{w-\mu_x^c}{v-n\mu_x^c}\right)\right\}^{n-2}dw;\quad v>n\mu_x^c\quad,\quad u>m\mu_y^c & \\ 0\;;\quad otherwise & \end{cases}$$

This implies

$$\tilde{R}_{\mu_x,\,\mu_y} = \begin{cases} (m-1)(n-1)\displaystyle\sum_{i=0}^{m-2}\sum_{j=0}^{n-2}\sum_{k=0}^{i+1}(-1)^{i+j}\binom{m-2}{i}\binom{n-2}{j}\binom{i+1}{k} \\ \times\dfrac{(\mu_x^c-\mu_y^c)^{i+1-k}(v-n\mu_x^c)^k}{(i+1)(u-m\mu_y^c)^{i+1}(j+k+1)}; \quad v>n\mu_x^c,\ u>m\mu_y^c \\ 0\ ; \quad otherwise \end{cases} . \qquad (4.11)$$

Hence, the lemma is proved.

BAYES ESTIMATOR OF R_{μ_x,μ_y}

In view of (2.5) and (2.7), the posterior distribution of θ, say $\pi_1(\theta|\underline{x})$, is

$$\pi_1(\theta|\underline{x}) = \frac{L_1(\underline{x}|\theta)\,h_1(\theta)}{\int_0^\infty L_1(\underline{x}|\theta)\,h_1(\theta)\,d\theta}$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^c - n\mu_x^c + \alpha\right)^{n+\beta}\exp\left\{-\dfrac{1}{\theta}\left(\sum_{i=1}^{n} x_i^c - n\mu_x^c + \alpha\right)\right\}}{\theta^{n+\beta+1}\Gamma(n+\beta)}; \quad \theta>0$$

(4.12)

Similarly, in view of (2.6) and (2.8), the posterior distribution of λ, say $\pi_2(\lambda|\underline{y})$ is

$$\frac{\left(\sum_{j=1}^{m} y_j^a - m\mu_y^a + \xi\right)^{m+\tau}\exp\left\{-\dfrac{1}{\lambda}\left(\sum_{j=1}^{m} y_j^a - m\mu_y^a + \xi\right)\right\}}{\lambda^{m+\tau+1}\Gamma(m+\tau)}; \quad \lambda>0 \qquad (4.13)$$

On using the posterior distributions obtained in (4.12) and (4.13), the Bayes estimator, say $R^*_{\mu_x,\mu_y}$ of the re-modeled reliability, R_{μ_x,μ_y} in (3.1) under $(SELF)$ is

$$R^*_{\mu_x,\,\mu_y} = E[R_{\mu_x,\mu_y} \mid \underline{x},\underline{y}] = \int_0^\infty\int_0^\infty R_{\mu_x,\,\mu_y}\,\pi_1(\theta|\underline{x})\,\pi_2(\lambda|\underline{y})\,d\theta\,d\lambda$$

$$=1-\left\{\begin{array}{l}(n+\beta)\left(\sum_{i=1}^{n}x_i^c-n\mu_x^c+\alpha\right)\left(\sum_{j=1}^{m}y_j^a-m\mu_y^a+\xi\right)\\ \times\int_{\mu_x^c}^{\infty}\dfrac{cx^{c-1}dx}{\left(\sum_{i=1}^{n}x_i^c-n\mu_x^c+\alpha+x^c-\mu_x^c\right)^{n+\beta+1}\left(\sum_{j=1}^{m}y_j^a-m\mu_y^a+\xi+x^a-\mu_y^a\right)^{m+\tau}}\end{array}\right.$$

(4.14)

The posterior variance of the Bayes estimate, $R^{*}_{\mu_x,\,\mu_y}$ is given by

$$V[R^{*}_{\mu_x,\,\mu_y}]=E[R^{2}_{\mu_x,\mu_y}\mid \underline{x},\underline{y})]-[E\{R_{\mu_x,\mu_y}\mid \underline{x},\underline{y}\}]^{2}, \tag{4.15}$$

where

$$E[R^{2}_{\mu_x,\mu_y}\mid \underline{x},\underline{y}]=\int_0^{\infty}\int_0^{\infty}R^{2}_{\mu_x,\mu_y}\,\pi_1(\theta|\underline{x})\,\pi_2\,(\lambda|\underline{y})\,d\theta\,d\lambda \tag{4.16}$$

Since, Bayes estimate $R^{*}_{\mu_x,\,\mu_y}$ in (4.14) cannot be obtained in closed form solution. Therefore, we use numerical integration and Lindley approximation approaches to obtain $R^{*}_{\mu_x,\,\mu_y}$.

LINDLEY APPROXIMATION APPROACH

Using Lindley approximation procedure, the Bayes estimator of re-modeled reliability in (3.1) can be written as

$$\begin{aligned}R^{*}_{\mu_x,\mu_y}&=u+\frac{(u_{11}\sigma_{11}+u_{22}\sigma_{22})}{2}+\rho_1u_1\sigma_{11}+\rho_2u_2\sigma_{22}\\&+\frac{(L_{30}u_1\sigma_{11}^2+L_{03}u_2\sigma_{22}^2+L_{21}u_2\sigma_{11}\sigma_{22}+L_{12}u_1\sigma_{11}\sigma_{22})}{2}\\&+\text{evaluated at }(\hat{\theta},\hat{\lambda}),\end{aligned} \tag{4.17}$$

where

$$u=u(\theta,\lambda)=R(\theta,\lambda)=1-\int_0^{\infty}\exp\left[-\left\{g+\left(\frac{(\theta g+\mu_x^c)^{a/c}-\mu_y^a}{\lambda}\right)\right\}\right]dg,$$

$$u_1 = \frac{du}{d\theta} = \left[\int_0^\infty \frac{a\, g(\theta g + \mu_x^c)^{\frac{a}{c}-1}}{c\lambda} \exp\left[-\left\{ g + \left(\frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda} \right) \right\} \right] dg \right]_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}} ,$$

$$u_2 = \frac{du}{d\lambda} = \left[-\int_0^\infty \frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda^2} \exp\left[-\left\{ g + \left(\frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda} \right) \right\} \right] dg \right]_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$u_{11} = \frac{d^2u}{d\theta^2} = \left[\int_0^\infty \frac{ag^2}{c\lambda} \left(\frac{a}{c} - 1 \right) (\theta g + \mu_x^c)^{\frac{a}{c}-2} \exp\left[-\left\{ g + \left(\frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda} \right) \right\} \right] dg \right]_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$- \left[\int_0^\infty \left(\frac{ag}{c\lambda} \right)^2 \left\{ (\theta g + \mu_x^c)^{\frac{a}{c}-1} \right\}^2 \exp\left[-\left\{ w + \left(\frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda} \right) \right\} \right] dg \right]_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$u_{22} = \frac{d^2u}{d\lambda^2} = \left[\int_0^\infty \frac{2(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda^3} \exp\left[-\left\{ g + \left(\frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda} \right) \right\} \right] dg \right]_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$- \left[\int_0^\infty \left\{ \frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda^2} \right\}^2 \exp\left[-\left\{ g + \left(\frac{(\theta g + \mu_x^c)^{a/c} - \mu_y^a}{\lambda} \right) \right\} \right] dg \right]_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$\sigma_{11} = \left(-\frac{d^2L}{d\theta^2} \right)^{-1} = \left\{ \frac{n}{\theta^2} - \frac{2}{\theta^3} \left(\sum_{i=1}^n x_i^c - n\mu_x^c \right) \right\}^{-1}_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$\sigma_{22} = \left(-\frac{d^2L}{d\lambda^2} \right)^{-1} = \left\{ \frac{m}{\lambda^2} - \frac{2}{\lambda^3} \left(\sum_{j=1}^m y_j^a - m\mu_y^a \right) \right\}^{-1}_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$L_{21} = \left(\frac{d^3 L}{d\theta^2 d\lambda} \right) = 0 = L_{12} = \left(\frac{d^3 L}{d\lambda^2 d\theta} \right)$$

$$L_{30} = \left(\frac{d^3 L}{d\theta^3} \right) = \left\{ \frac{6}{\theta^4} \left(\sum_{i=1}^{n} x_i^c - n\mu_x^c \right) - \frac{2n}{\theta^3} \right\}_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$L_{03} = \left(\frac{d^3 L}{d\lambda^3} \right) = \left\{ \frac{6}{\lambda^4} \left(\sum_{j=1}^{m} y_j^a - m\mu_y^a \right) - \frac{2m}{\lambda^3} \right\}_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$\rho_1 = \frac{d \log\{g(\theta,\lambda)\}}{d\theta} = \left\{ \frac{\alpha}{\theta^2} - \frac{(\beta+1)}{\theta} \right\}_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

$$\rho_2 = \frac{d \log\{g(\theta,\lambda)\}}{d\lambda} = \left\{ \frac{\xi}{\lambda^2} - \frac{(\tau+1)}{\lambda} \right\}_{\substack{\theta=\hat{\theta} \\ \lambda=\hat{\lambda}}}$$

and

$$L = \log L(\underline{x}, \underline{y} | \theta, \lambda) = n \log c_1 - n \log \theta - \frac{1}{\theta} \left(\sum_{i=1}^{n} x_i^c - n\mu_x^c \right)$$

$$+ m \log a_1 - m \log \lambda - \frac{1}{\lambda} \left(\sum_{j=1}^{m} y_j^a - m\mu_y^a \right)$$

$$\log\{g(\theta,\lambda)\} = \beta \log \alpha - \frac{\alpha}{\theta} - \log \Gamma(\beta) - (\beta+1)\log\theta + \tau \log \xi$$

$$- \frac{\xi}{\lambda} - \log\Gamma(\tau) - (\tau+1)\log\lambda .$$

5. NUMERICAL STUDY AND CONCLUSIONS

Simulated samples of sizes $n = 30$ and $m = 30$ were drawn from distributions given in (2.3) and (2.4) respectively. On using the simulated sample information and the relevant expressions, the various estimates (Classical and Bayes both) of R_{μ_x,μ_y} for fixed μ_y, θ, $E(\theta)$, λ, $E(\lambda)$, $a = c$ and varying μ_x have been summarized in table-1.

Similarly, various estimates (Classical and Bayes both) of R_{μ_x,μ_y} for fixed μ_x, θ, $E(\theta)$, λ, $E(\lambda)$, $a = c$ and varying μ_y have been listed in table-2. The entries given in the parentheses represent the corresponding *MSE* of the estimates.

From tables-1 and 2, it is observed that

- True reliability and its various estimates ($\hat{R}_{\mu_x,\mu_y}$, $\tilde{R}_{\mu_x,\mu_y}$, $R^*_{\mu_x,\mu_y}$ and $R^\bullet_{\mu_x,\mu_y(Lindley)}$) tend to be low in case $\mu_x < \mu_y$, i.e., the stress experienced by the system is greater than the guaranteed strength.
- True reliability and its various estimates ($\hat{R}_{\mu_x,\mu_y}$, $\tilde{R}_{\mu_x,\mu_y}$, $R^*_{\mu_x,\mu_y}$ and $R^\bullet_{\mu_x,\mu_y(Lindley)}$) tend to be high when system is guaranteed for higher strength with lower stress, i.e., $\mu_y < \mu_x$.
- As expected, the *UMVUE* of R_{μ_x,μ_y} is more efficient than other estimates like $\hat{R}_{\mu_x,\mu_y}$, $R^*_{\mu_x,\mu_y}$ and $R^\bullet_{\mu_x,\mu_y(Lindley)}$.
- The reliability and its various estimates increase (decrease) uniformly as the value of μ_x increases (μ_y increases).

Thus, for analyzing the trends observed above of the theoretical developments, the system designer can achieve a trade-off between the intended reliability with guaranteed strength subjected to some at least stress.

Table 5.1: Classical and Bayes estimates of R_{μ_x,μ_y} for fixed $\mu_y = 6.5$, $\theta = E(\theta) = 4.5$, $\lambda = E(\lambda) = 2.0$, a = c = 0. 5 and varying μ_x

μ_x	R_{μ_x,μ_y}	$\hat{R}_{\mu_x,\mu_y}$	$\tilde{R}_{\mu_x,\mu_y}$	$R^*_{\mu_x,\mu_y}$	$R^\bullet_{\mu_x,\mu_y}(Lindley)$
0.5	0.226	0.382653 (0.26544)	0.237303 (0.00891)	0.156786 (0.015904)	0.367772 (0.01982)
6.5	0.692	0.631324 (0.1474)	0.651903 (0.004269)	0.711437 (0.002578)	0.635844 (0.0032)
12.5	0.812	0.757375 (0.166)	0.867057 (0.00125)	0.811008 (0.002135)	0.760609 (0.0027)
18.5	0.872	0.819781 (0.167)	0.915 (0.000145)	0.866339 (0.001652)	0.822101 (0.0025)
24.5	0.91	0.850415 (0.1364)	0.92 (0.000311)	0.884813 (0.001516)	0.852768 (0.003)

TABLE 5.2: Classical and Bayes estimates of R_{μ_x,μ_y} for fixed $\mu_x = 12.5$, $\theta = E(\theta) = 4.5$, $\lambda = E(\lambda) = 2.0$, $a = c = 0.5$ and varying μ_y

μ_y	R_{μ_x,μ_y}	$\hat{R}_{\mu_x,\mu_y}$	$\tilde{R}_{\mu_x,\mu_y}$	$R^{*}_{\mu_x,\mu_y}$	$R^{\bullet}_{\mu_x,\mu_y}$ $(Lindley)$
0.5	0.925	0.935635 (0.044)	0.965417 (0.000241)	0.898176 (0.001409)	0.931545 (0.00004)
6.5	0.812	0.823 (0.0411)	0.814031 (0.001854)	0.803031 (0.002252)	0.828037 (0.000255)
12.5	0.692	0.614076 (0.075)	0.689884 (0.003665)	0.677188 (0.00294)	0.626301 (0.0044)
18.5	0.548	0.474602 (0.060)	0.57817 (0.021915)	0.611749 (0.003027)	0.491231 (0.00331)
24.5	0.375	0.278761 (0.065)	0.372575 (0.017927)	0.45142 (0.00562)	0.297377 (0.00618)

Acknowledgement

The authors extend their sincere thanks to the UGC, New Delhi, India for providing the financial assistance for carrying out the research work and learned reviewer for helpful suggestions and comments for improving the manuscript.

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Received : 16-01-2008

Revised: 29-05-2010

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