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ON TWO BIVARIATE GEOMETRIC DISTRIBUTIONS

K.S. Bhanu and M.N. Deshpande

ABSTRACT

A bivariate geometric distribution where both the marginals are geometric distributions has been studied in this article. Two such types of bivariate distributions are explained. The properties and genesis of the same are discussed.

1. INTRODUCTION

A bivariate geometric distribution is one where both the marginal distributions are geometric distributions. In this article, we show that there exist two such types of bivariate distributions. The properties and genesis of the same are discussed.

A random variable X is a geometric variate, if it has the following probability mass function (pmf)

$$p(x) = pq^x, \quad x = 0,1,2,\dots; \quad 0 < p < 1 \text{ and } q = 1 - p.$$

The moment generating function (mgf) of this distribution is given by $p(1-qe^t)^{-1}$. The mean and variance of the distribution are given by q/p and q/p^2 respectively.

2. FIRST FAMILY

Random variables X and Y belong to Type I bivariate geometric distribution, if its joint pmf is given by

$$p(x,y) = \frac{p_1 p_2}{p_1 + p_2 - p_1 p_2} \binom{x+y}{x} \left[\frac{(1-p_1)\, p_2}{p_1 + p_2 - p_1 p_2} \right]^x \left[\frac{(1-p_2)\, p_1}{p_1 + p_2 - p_1 p_2} \right]^y,$$

$$0 \le x, y; \quad 0 < p_1, p_2 < 1 \qquad (2.1)$$

The mgf of the above bivariate distribution is obtained as

$$M_{X,Y}(t_1,t_2) = E[e^{(t_1 X + t_2 Y)}]$$

$$= \frac{p_1 p_2}{[p_1 + p_2 - p_1 p_2 - p_2(1-p_1)e^{t_1} - p_1(1-p_2)e^{t_2}]}. \qquad (2.2)$$

It follows that the marginal distributions of X and Y are geometric with parameters p_1 and p_2 respectively.

Further,

$$Cov(X,Y) = \frac{(1-p_1)(1-p_2)}{p_1 p_2}$$

and correlation coefficient between X and Y is given by

$$\rho(X,Y) = \sqrt{(1-p_1)(1-p_2)} \ . \tag{2.3}$$

We note that X and Y are always positively correlated.

This family is discussed in (Kocherlakota and Kocherlakota (1992) on page 125), where the parameters are slightly changed. However, we have presented the same because we present below another type of a similar family. Further, we discuss a simple experiment as a result of which both the families appear.

3. SECOND FAMILY

Random variables X and Y belong to Type II bivariate geometric distribution, if it has the following joint pmf

$$p(x,y) = p_1 \binom{x}{y} \left[1 - \frac{p_1}{p_2}\right]^x \left[\frac{p_1(1-p_2)}{p_2 - p_1}\right]^y,$$

$$0 \le y \le x \text{ and } 0 < p_1 < p_2 < 1 \tag{3.1}$$

The mgf of the above bivariate distribution is obtained as

$$M_{X,Y}(t_1,t_2) = E[e^{(t_1 X + t_2 Y)}]$$

$$= \frac{p_1 p_2}{[p_2 - (p_2 - p_1)e^{t_1} - p_1(1-p_2)e^{(t_1+t_2)}]} \tag{3.2}$$

It follows from this that the marginal distributions of X and Y are geometric with parameters p_1 and p_2 respectively.

Further,

$$Cov(X,Y) = \frac{(1-p_2)}{p_1 p_2}$$

and correlation coefficient between X and Y is

$$\rho(X,Y) = \sqrt{\frac{(1-p_2)}{(1-p_1)}} \ . \tag{3.3}$$

In this case also X and Y are positively correlated.

4. GENESIS

We discuss one simple experiment which gives rise to both the above bivariate distributions.

Experiment: Let θ be the probability of getting a head (H) when a coin is tossed. The coin is tossed till we get three successive heads. Let the random variables X, Y and Z denote respectively, the number of tails (T), number of single heads (H) and the number of double heads (HH) obtained before the tossing of the coin is stopped. Then the probability of getting a string consisting of x $T's$, y $H's$ and z $HH's$ equals $(1-\theta)^x \theta^y (\theta^2)^z \theta^3$ i.e. it equals $(1-\theta)^x \theta^{y+2z+3}$.

The total number of such strings will be $\dfrac{x!}{y!z!(x-y-z)!}$.

This can be seen as follows:

The sequence of heads and tails terminates when we get HHH. The outcome immediately preceding this string of HHH is obviously a T. The x tails are intermixed with y (H) and z (HH). The y $H's$ can occur before x tails in $\binom{x}{y}$ ways and z $HH's$ can occur among the remaining $(x-y)$ places in $\binom{x-y}{z}$ ways. Hence the total number of places in which x $T's$, y $H's$ and z $HH's$ can occur are

$$\binom{x}{y}\binom{x-y}{z} = \frac{x!}{y!z!(x-y-z)!}$$

Thus the joint pmf of X, Y and Z is

$$p(x,y,z) = \frac{x!}{y!z!(x-y-z)!}(1-\theta)^x \theta^{y+2z+3} \tag{4.1}$$

5. JOINT DISTRIBUTION OF Y AND Z

The joint distribution of Y and Z can be obtained as follows:

$$p(y,z) = \sum_x p(x,y,z)$$

$$= \theta^{(y+2x+3)} (1-\theta)^{(y+z)} \binom{y+z}{z} \sum_x \left\{ \binom{x}{y+z} (1-\theta)^{(x-y-z)} \right\}$$

$$= \theta^{(y+2x+3)} (1-\theta)^{(y+z)} \binom{y+z}{z} \left(\frac{1}{\theta^{(y+z+1)}} \right)$$

$$= \binom{y+z}{z} \theta^{(z+2)} (1-\theta)^{(y+z)} , \quad 0 \le y, z$$

which can be written as

$$p(y,z) = \binom{y+z}{z} \theta^2 [\theta(1-\theta)]^z (1-\theta)^y \tag{5.1}$$

Letting $p_1 = \theta$ and $p_2 = \dfrac{\theta^2}{(1-\theta+\theta^2)}$ in (2.1) and simplifying, we can obtain (5.1).

It may be noted that $p_2 < p_1$ as given in (3.1). One interesting result in this case is that $E(Y) = Var(Z)$ for any value of θ.

6. JOINT DISTRIBUTION OF *X* AND *Y*

The joint distribution of X and Y can be obtained in a similar fashion as follows:

$$p(x,y) = \sum_z p(x,y,z)$$

$$= \binom{x}{y} \theta^{(y+3)} (1-\theta)^x \sum_z \left\{ \binom{x-y}{z} (\theta^2)^z \right\}$$

$$= \binom{x}{y} \theta^{(y+3)} (1-\theta)^x (1+\theta^2)^{x-y}$$

$$= \binom{x}{y} \theta^3 [(1-\theta)(1+\theta^2)]^x \left(\frac{\theta}{1+\theta^2} \right)^y , \quad 0 \le y \le x. \tag{6.1}$$

Again letting, $p_1 = \theta^3$ and $p_2 = \dfrac{\theta^2}{(1-\theta+\theta^2)}$ in (3.1) and simplifying, it can be shown that (3.1) coincides with (6.1).

We can, therefore, say that in the above coin tossing experiment, the joint distribution of the random variables (Y, Z) belongs to the family of Type I bivariate geometric distribution and the joint distribution of the random variables (X, Y) belongs to type II bivariate geometric family. In a similar manner, the joint distribution of the random variables (X, Z) can also be shown to belong to type II bivariate geometric family.

REFERENCES

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Department of Statistics,
Institute of Science, Nagpur-440 001, India
email: bhanu_105@yahoo.com
dpratap_ngp @sancharnet.in