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RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF DUAL GENERALIZED ORDER STATISTICS FROM EXPONENTIATED WEIBULL DISTRIBUTION

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ABSTRACT

In Burkschat *et al.* (2003) dual generalized order statistics have been proposed that enables a common approach to descendingly ordered random variables like reversed ordered order statistics, lower $k-$th records and lower Pfeifer records. With this definition we give recurrence relations for single and product moments of dual generalized order statistics from exponentiated Weibull distribution. Further some related results and particular cases are discussed.

1. INTRODUCTION

The concept of generalized order statistics (gos) was introduced by Kamps (1995) as below:

Let $F()$ be an absolutely continuous distribution function (df) with probability density function (pdf) $f()$. Further, let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0 \quad \forall \quad r \in \{1, 2, \cdots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \cdots, n$ are called gos if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\Re^n$.

The model of gos contains as special cases, order statistics $(m = 0, k = 1)$, $k-$th record values $(m_1 = m_2 = \cdots = m_{n-1} = -1$, ie. $\gamma_i = k$, $k \in N)$, sequential order statistics $(\gamma_i = (n-i+1)\alpha_i;\ \alpha_1, \alpha_2, \cdots, \alpha_n > 0)$, order statistics with non-integral sample size $(\gamma_i = \alpha - i + 1,\ \alpha > 0)$. But when $F()$ is an inverse distribution function, we need a concept of dual generalized order statistics, which was introduced by Burkschat *et al.* (2003) as follows:

Let $n \in N$, $k \geq 1$, $m \in R$, be the parameters such that $\gamma_r = k + n - r + M_r > 0$, $M_r = \sum_{j=r}^{n-1} m_j \quad \forall \quad 1 \leq r \leq n$. By the dual *gos* from an absolutely continuous distribution function $F()$ with density function $f()$ we mean random variables $X'(1, n, \tilde{m}, k), \cdots, X'(n, n, \tilde{m}, k)$ having joint density function of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f((x_i)\right)[F(x_n)]^{k-1} f(x_n) \tag{1.2}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0)$.

Here we may consider two cases:

Case I: $m_i = m_j = m$, $\quad i, j = 1, 2, \cdots, n-1$.

The density function of $r-$th dual generalized order statistic is given by

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)). \tag{1.3}$$

The joint density function of $r-$th and $s-$th dual generalized order statistics is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)\, g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y),$$

$$\alpha \leq y < x \leq \beta, \tag{1.4}$$

where

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\log x, & m = 1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \qquad x \in [0,1).$$

Case II: $\gamma_i \neq \gamma_j$, $\quad i \neq j$, $\quad i, j = 1, 2, \cdots, n-1$.

The *pdf* of $r-$th dual generalized order statistic is given by

$$f_{X'(r,n,\tilde{m},k)}(x) = C_{r-1}\, f(x)\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1} \tag{1.5}$$

and the joint *pdf* of $r-$th and $s-$th dual generalized order statistics is

$$f_{X'(r,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i}$$

$$\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)},$$

$$\alpha \le y < x \le \beta, \tag{1.6}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + M_i,$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \le i \le r \le n$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \le i \le s \le n.$$

A random variable X is said to have exponentiated Weibull distribution (Mudholkar *et al.*, 1995) if its *pdf* is given by

$$f(x) = \tau\theta\lambda^{\theta} x^{\theta-1} e^{-(\lambda x)^{\theta}} [1 - e^{-(\lambda x)^{\theta}}]^{\tau-1}, \; x > 0, \; \lambda > 0, \; \theta > 0, \; \tau > 0 \tag{1.7}$$

and the corresponding *df* is

$$F(x) = [1 - e^{-(\lambda x)^{\theta}}]^{\tau}. \tag{1.8}$$

Therefore, in view of (1.7) and (1.8), we have

$$F(x) = \frac{1}{\tau\theta\lambda^{\theta}} x^{1-\theta} [e^{(\lambda x)^{\theta}} - 1] f(x). \tag{1.9}$$

For more details on this distribution and its application one may refer to Mudholkar and Hutson (1996) and Nassar and Eissa (2003).

In this paper we have exploited the relation (1.9) to obtain the recurrence relations for single and product moments of dual generalized order statistics for the distribution as given in (1.7).

2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

Before coming to the main results we shall prove the following lemma:

Case I: $m_i = m_j = m, \quad i, j = 1, 2, \cdots, n-1$.

Lemma 2.1: For $2 \le r \le n$, $n \ge 2$ and $k = 1, 2, \cdots$.

$$E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n,m,k)]$$

$$= -\frac{j\, C_{r-1}}{\gamma_r\,(r-1)!}\int_{\alpha}^{\beta} x^{j-1}[F(x)]^{\gamma_r}\, g_m^{r-1}(F(x))\,dx. \tag{2.1}$$

Proof: We have,

$$E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n,m,k)]$$

$$= \frac{C_{r-1}}{(r-1)!}\int_{\alpha}^{\beta} x^{j}[F(x)]^{\gamma_r} f(x)\, g_m^{r-2}(F(x))\left\{\frac{g_m(F(x))}{F(x)} - \frac{(r-1)[F(x)]^m}{\gamma_r}\right\}dx. \tag{2.2}$$

Let $h(x) = \dfrac{1}{\gamma_r}[F(x)]^{\gamma_r}\, g_m^{r-1}(F(x))$

$$h'(x) = [F(x)]^{\gamma_r} f(x)\, g_m^{r-2}(F(x))\left\{\frac{g_m(F(x))}{F(x)} - \frac{(r-1)[F(x)]^m}{\gamma_r}\right\}.$$

Thus RHS of (2.2) reduces to

$$\frac{C_{r-1}}{(r-1)!}\int_{\alpha}^{\beta} x^{j}\, h'(x)\, dx$$

and hence integrating by parts, we get the result.

Lemma 2.2: For $2 \le r \le n$, $n \ge 2$ and $k = 1, 2, \cdots$.

i) $E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n-1,m,k)]$

$$= -\frac{j\, C_{r-1}}{\gamma_1\,(r-1)!}\int_{\alpha}^{\beta} x^{j-1}[F(x)]^{\gamma_r}\, g_m^{r-1}(F(x))\,dx. \tag{2.3}$$

ii) $E[X'^{j}(r-1,n,m,k)] - E[X'^{j}(r-1,n-1,m,k)]$

$$= \frac{(m+1)\, j\, C_{r-2}}{\gamma_1\,(r-2)!}\int_{\alpha}^{\beta} x^{j-1}[F(x)]^{\gamma_r}\, g_m^{r-1}(F(x))\,dx. \tag{2.4}$$

Proof: Proof follows on the lines of Lemma 2.1.

Theorem 2.1: For the distribution given in (1.8) and for $2 \le r \le n$, $n \ge 2$ and $k = 1, 2, \cdots$

$$E[X'^j(r,n,m,k)] - E[X'^j(r-1,n,m,k)]$$

$$= -\frac{j}{\gamma_r \tau \theta \lambda^\theta}\left\{E[\psi\{X'(r,n,m,k)\}] - E[X'^{j-\theta}(r,n,m,k)]\right\}, \quad (2.5)$$

where $\psi(x) = x^{j-\theta} e^{(\lambda x)^\theta}$.

Proof: From (1.9) and (2.1), we have

$$E[X'^j(r,n,m,k)] - E[X'^j(r-1,n,m,k)]$$

$$= -\frac{j\, C_{r-1}}{\gamma_r (r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r - 1}\left\{\frac{x^{1-\theta}[e^{(\lambda x)^\theta} - 1]}{\tau \theta \lambda^\theta}\right\} f(x)\, g_m^{r-1}(F(x))\, dx$$

$$= -\frac{j}{\gamma_r \tau \theta \lambda^\theta}\left\{\frac{C_{r-1}}{(r-1)!}\int_0^\infty \psi(x)[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\, dx\right.$$

$$\left. - \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-\theta}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\, dx\right\}$$

and hence the result.

Remark 2.1: For $m = 0$, $k = 1$, the recurrence relation for dual generalized order statistics reduces to the recurrence relation of ordinary order statistics as

$$E(X'^j_{n-r+1:n}) - E(X'^j_{n-r+2:n})$$

$$= -\frac{j}{\tau \theta \lambda^\theta (n-r+1)}\left\{E[\psi(X'_{n-r+1:n})] - E(X'^{j-\theta}_{n-r+1:n})\right\}. \quad (2.6)$$

Remark 2.2: The recurrence relation for single moment of $k-$th lower record statistics $(m = -1)$ will be

$$E(X'^j_{n-r+1})^k - E(X'^j_{n-r+2})^k$$

$$= -\frac{j}{\tau \theta \lambda^\theta k}\left\{E[\psi(X'_{n-r+1})^k] - E(X'^{j-\theta}_{n-r+1})^k\right\}. \quad (2.7)$$

Remark 2.3: For $m = 0$ and $k = \alpha - n + 1$, $\alpha \in \Re_+$, the recurrence relation for single moments of dual order statistics with non-integral sample size is

$$E(X'^{j}_{\alpha-r+1:\alpha}) - E(X'^{j}_{\alpha-r+2:\alpha})$$

$$= -\frac{j}{\tau\theta\lambda^{\theta}(\alpha-r+1)}\left\{E[\psi(X'_{\alpha-r+1:\alpha})] - E(X'^{j-\theta}_{\alpha-r+1:\alpha})\right\}. \tag{2.8}$$

Remark 2.4: For $m=\alpha-1$ and $k=\alpha$, the recurrence relation for sequential order statistics is

$$E[X'^{j}(r,n,\alpha-1,\alpha)] - E[X'^{j}(r-1,n,\alpha-1,\alpha)]$$

$$= -\frac{j}{\tau\theta\lambda^{\theta}\alpha(n-r+1)}\left\{E[\psi\{X'(r,n,\alpha-1,\alpha)\}] - E[X'^{j-\theta}(r,n,\alpha-1,\alpha)]\right\}. \tag{2.9}$$

Theorem 2.2: For the distribution given in (1.8) and for $2 \le r \le n$, $n \ge 2$ and $k = 1,2,\cdots$.

i) $E[X'^{j}(r,n,m,k)] - E[X'^{j}(r-1,n-1,m,k)]$

$$= -\frac{j}{\gamma_1\tau\theta\lambda^{\theta}}\left\{E[\psi\{X'(r,n,m,k)\}] - E[X'^{j-\theta}(r,n,m,k)]\right\}. \tag{2.10}$$

ii) $E[X'^{j}(r-1,n,m,k)] - E[X'^{j}(r-1,n-1,m,k)]$

$$= \frac{(m+1)(r-1)\,j}{\gamma_1\gamma_r\tau\theta\lambda^{\theta}}\left\{E[\psi\{X'(r,n,m,k)\}] - E[X'^{j-\theta}(r,n,m,k)]\right\}. \tag{2.11}$$

Proof: Results can be established in view of Lemma 2.2 and (1.9).

Case II: $\gamma_i \ne \gamma_j$, $i \ne j$, $i, j = 1,2,\cdots,n-1$.

Lemma 2.3: For $2 \le r \le n$, $n \ge 2$ and $k = 1,2,\cdots$.

$$E[X'^{j}(r,n,\tilde{m},k)] - E[X'^{j}(r-1,n,\tilde{m},k)]$$

$$= -j\,C_{r-2}\sum_{i=1}^{r} a_i(r)\int_{\alpha}^{\beta} x^{j-1}[F(x)]^{\gamma_r}\,dx. \tag{2.12}$$

Proof: We have,

$$E[X'^{j}(r,n,\tilde{m},k)] - E[X'^{j}(r-1,n,\tilde{m},k)]$$

$$= C_{r-2}\sum_{i=1}^{r} a_i(r)\int_{\alpha}^{\beta} x^{j} f(x)\gamma_i[F(x)]^{\gamma_i-1}dx. \tag{2.13}$$

Let $h(x) = [F(x)]^{\gamma_i}$

$$h'(x) = \gamma_i [F(x)]^{\gamma_i - 1} f(x) .$$

Thus *RHS* of (2.13) reduces to

$$C_{r-2} \sum_{i=1}^{r} a_i(r) \int_{\alpha}^{\beta} x^j \, h'(x) \, dx . \tag{2.14}$$

Now integrating in (2.14) by parts and noting that $\sum_{i=1}^{r} a_i(r) = 0$, $r \ge 2$, we have the result.

Theorem 2.3: For the distribution given in (1.8) and $2 \le r \le n$, $n \ge 2$ and $k = 1, 2, \cdots$.

$$E[X'^{j}(r,n,\tilde{m},k)] - E[X'^{j}(r-1,n,\tilde{m},k)]$$

$$= -\frac{j}{\gamma_r \, \tau \theta \lambda^{\theta}} \left\{ E[\psi\{X'(r,n,\tilde{m},k)\}] - E[X'^{\,j-\theta}(r,n,\tilde{m},k)] \right\} . \tag{2.15}$$

Proof: Proof follows on the lines of Theorem 2.1 using (1.9) and lemma 2.3.

Remark 2.5: Theorem 2.1 can be deduced from Theorem 2.3 by replacing $\tilde{m}$ with m, $m \ne -1$. Remaining relations for case II, $(\gamma_i \ne \gamma_j)$ can be established by replacing m with $\tilde{m}$ in Theorem 2.2.

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

Case I: $m_i = m_j = m$, $i, j = 1, 2, \cdots, n-1$.

Lemma 3.1: For $1 \le r < s \le n-1$, $n \ge 2$ and $k = 1, 2, \cdots$.

$$E[X'^{i}(r,n,m,k) \; X'^{j}(s,n,m,k)] - E[X'^{i}(r,n,m,k) \; X'^{j}(s-1,n,m,k)]$$

$$= -\frac{j \, C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_{\alpha}^{\beta} \int_{\alpha}^{x} x^i y^{j-1} [F(x)]^m f(x) \, g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x)]^{s-r-1} [F(y)]^{\gamma_s} \, dy \, dx , \quad x > y \tag{3.1}$$

Proof: We have,

$$E[X'^{i}(r,n,m,k) \; X'^{j}(s,n,m,k)] - E[X'^{i}(r,n,m,k) \; X'^{j}(s-1,n,m,k)]$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{\alpha}^{\beta} \int_{\alpha}^{x} x^i y^j [F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x)]^{s-r-2} [F(y)]^{\gamma_s - 1} f(y)$$

$$\times \left\{ [h_m(F(y)) - h_m(F(x))] - \frac{(s-r-1)}{\gamma_s} [F(y)]^{m+1} \right\} dy\, dx . \tag{3.2}$$

Let $h(x, y) = \frac{1}{\gamma_s} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s}$ (3.3)

$$\frac{\partial}{\partial y} h(x, y) = [h_m(F(y)) - h_m(F(x))]^{s-r-2} \gamma_s [F(y)]^{\gamma_s - 1} f(y)$$

$$\times \left\{ [h_m(F(y)) - h_m(F(x))] - \frac{(s-r-1)}{\gamma_s} [F(y)]^{m+1} \right\} . \tag{3.4}$$

Taking into account the value of (3.4) in (3.2), we get

$$E[X'^{i}(r,n,m,k)\ X'^{j}(s,n,m,k)] - E[X'^{i}(r,n,m,k)\ X'^{j}(s-1,n,m,k)]$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{\alpha}^{\beta} x^i [F(x)]^m f(x) g_m^{r-1}(F(x)) \left(\int_{\alpha}^{x} y^j \frac{\partial}{\partial y} h(x, y)\, dy \right) dx .$$

. (3.5)

Now in view of (3.3),

$$\int_{\alpha}^{x} y^j \frac{\partial}{\partial y} h(x, y)\, dy = -\frac{j}{\gamma_s} \int_{\alpha}^{x} y^{j-1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} dy$$

(3.6)

and hence the result.

Theorem 3.1: For the distribution given in (1.8) and for $1 \le r < s \le n-1$, $n \ge 2$ and $k = 1, 2, \cdots$

$$E[X'^{i}(r,n,m,k)\ X'^{j}(s,n,m,k)] - E[X'^{i}(r,n,m,k)\ X'^{j}(s-1,n,m,k)]$$

$$= -\frac{j}{\gamma_s \tau \theta \lambda^{\theta}} \{ E[\psi\{X'(r,n,m,k),\ X'(s,n,m,k)\}]$$

$$- E[X'^{i}(r,n,m,k) X'^{j-\theta}(s,n,m,k)] \} , \tag{3.7}$$

where $\psi(x,y) = x^i y^{j-\theta} e^{(\lambda x)^\theta}$.

Proof: The result can be established in view of (1.9) and lemma 3.1.

Remark 3.1: Under the assumption given in Theorem 3.1 with $k=1$, $m=0$, we get the recurrence relation for product moments of dual order statistics and at $m=-1$, we have the recurrence relations for product moments of dual $k-$th record values.

Case II: $\gamma_i \neq \gamma_j$, $i \neq j$, $i, j = 1, 2, \cdots, n-1$.

Results can be established by replacing m with $\tilde{m}$.

Note: For $\theta = 1$, $\tau = 1$ the distribution reduces to exponential distribution.

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