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STOCHASTIC FRACTIONAL PROGRAMMING IN TRANSSHIPMENT PROBLEM

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ABSTRACT

The present paper analyses a transshipment problem whose objective function is fractional where the demand considered is uncertain. The objective here is to maximize the net expected revenue per unit transportation cost, i.e., the total expected revenue minus the transportation and transshipments cost. The stochastic transshipment problem is converted to an equivalent deterministic transportation problem. An algorithm is developed for the deterministic transportation problem and is numerically illustrated.

1. INTRODUCTION

Orden (1956) proposed a generalized transportation model in which transshipment through intermediate points is permitted. The purpose of this paper is to study the transshipment type fractional programming problem in which the parameters of only the numerator are affected by stochastic uncertainties with known probability distributions. The objective is to maximize the net expected revenue which is defined as the total expected revenue minus the sum of transportation and transshipment costs. In this paper transshipment problem that often occurs in the distribution system of the national department store chain is considered, treating the demands as uncertain. The algorithm developed for the resulting deterministic problem, itself is the outcome of the basic result that for linear fractional programming the absolute minimum occurs at a basic feasible solution. The technique applied by Ferguson and Dantzig (1956) and Garwin (1963) are used for dealing the problems under stochastic environment.

2. NOTATIONS AND THE FORMULATION OF THE PROBLEM

We consider a transshipment problem with m sources and n sinks numbered as $1, 2, \cdots, m$ and n sinks numbered as $m+1, m+2, \cdots, m+n$.

Let

$a_i =$ the quantity available at source $i = 1, 2, \cdots, m$

$d_j =$ the quantity demanded at sink $j = m+1, m+2, \cdots, m+n$

$x_{ij} =$ the quantity shipped from station i to j $(i, j = 1, 2, \cdots, m+n)$

$c_{ij} =$ the per unit shipment cost from station i to j $(i, j = 1, 2, \cdots, m+n)$

$u_i =$ quantity transshipped at the station i $(i = 1, 2, \cdots, m+n)$

$c_i =$ per unit transshipment cost (including unloading, reloading, and storage etc.) at the station i $(i = 1, 2, \cdots, m+n)$.

The problem is to determine x_{ij} so as to minimize the total cost of transportation and transshipment. It may be mathematically stated as under:

Problem P₁: Find x_{ij} so as to

$$\text{Minimize} \quad Z = \sum_{i=1}^{m+n}{}^{*} \sum_{j=1}^{m+n}{}^{*} c_{ij} x_{ij} + \sum_{i=1}^{m+n} c_i u_i \tag{2.1}$$

subject to

$$\sum_{j=1}^{m+n}{}^{*} x_{ij} = \begin{cases} a_i + u_i & i = 1, 2, \cdots, m \\ u_i & i = m+1, \cdots, m+n \end{cases} \tag{2.2}$$

$$\sum_{i=1}^{m+n}{}^{*} x_{ij} = \begin{cases} u_j & j = 1, 2, \cdots, m \\ d_j + u_j & j = m+1, \cdots, .m+n \end{cases} \tag{2.3}$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j \tag{2.4}$$

$\sum_{j=1}^{m+n}{}^{*}$ indicates that the term $j = i$ is excluded from the sum. The constraints (2.2) implies that the total quantity that leaves the source i $(= 1, 2, \cdots, m)$ is equal to the quantity available plus the quantity transshipped and the total quantity that leaves the sink i $(= m+1, m+2, \cdots, m+n)$ is equal to the quantity transshipped. Similarly constraints (2.3) implies that the total quantity that arrives at a source i $(= 1, 2, \cdots, m)$ is equal to the quantity that source transships and the total arriving at a sink is equal to the demand at that sink plus the quantity that the sink transships. Constraints (2.4) are the usual nonnegative restrictions. Here u_i are unknown, so we impose an upper bound u_0 (say), on the amount that can be transshipped at any point, so that

$$u_i = u_o - x_{ii}, \quad i = 1, 2, \cdots, m, \tag{2.5}$$

where x_{ii} is a nonnegative slack. After substituting (2.5) in (2.1) to (2.3) and on simplifying the original transshipment problem P_1 is reduced to the following genuine transportation type linear programming problem.

Problem P$_2$:

Minimize $Z = \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} c_{ij} x_{ij} + \sum_{i=1}^{m+n} c_i u_i$

subject to

$$\sum_{j=1}^{m+n} x_{ij} = \begin{cases} a_i + u_o & i = 1, 2, \cdots, m \\ u_o & i = m+1, \cdots, m+n \end{cases} \quad (2.6)$$

$$\sum_{i=1}^{m+n} x_{ij} = \begin{cases} u_o & j = 1, 2, \cdots, m \\ d_j + u_o & j = m+1, \cdots, m+n \end{cases} \quad (2.7)$$

$$x_{ij} \geq 0, \quad (2.8)$$

where $c_{ii} = -c_i$, and the asterisk (*) on the summations has now been disappeared. As $u_i \geq 0$, we must have $x_{ii} \leq u_o$, which is guaranteed by equations (2.6) – (2.7), because any x_{ii} will always appear in one equation that has u_o on the right hand side. Here, the upper bound u_o can interpreted as the size of a fictitious stockpile at each source and sink which is large enough to take care of all transshipments. Assume initially a value for u_o which is sufficiently large to ensure that all x_{ii} will be in the optimal basis. Such a value can be easily found as the volume of goods transshipped at any point cannot exceed the total volume of goods produced (or received). Hence, set,

$$u_o = \sum_{i=1}^{m} a_i \quad (2.9)$$

which ensures that u_0 is not limiting. The unused stockpile at the station $i = 1, 2, \cdots, m+n$, if any, will be absorbed in the slack x_{ii}.

3. PROBLEM REFORMULATION UNDER STOCHASTIC ENVIRONMENT

Till now, we have treated the demand d_j as uncertain as if they are fixed constraints. However, for our study, we assume d_j as an independent discrete random variable with known probability distributions. Let the unit selling price of the product shipped at $j-$th destination be s_j. Let the per unit loss due to pilferage etc on the route (i, j) be ρ_{ij}. The objective function can be written as the objective is to maximize the net expected revenue (minimize Z), that is defines as the total expected revenue minus the loss in transshipment. To take care of the randomness of demands, instead of minimizing the total cost, we take our objective as the maximization of the net expected revenue (i.e., total expected revenue minus transshipment and transportation costs). Here $\phi_j(s_j, y_j)$ is an unknown function that describes the expected revenue from destination j if a total of y_j unit is shipped to this destination.

Problem P$_3$:

$$\text{Min} \quad Z = \frac{\sum_{i=1}^{m+n}\sum_{j=1}^{m+n} \rho_{ij}\, x_{ij} - \sum_{j=1}^{m+n}\phi_j(s_j, y_j)}{\sum_{i=1}^{m+n}\sum_{j=1}^{m+n} c_{ij}\, x_{ij} - \sum_{i=1}^{m+n} c_i\, u_o} \tag{3.1}$$

subject to

$$\sum_{j=1}^{m+n} x_{ij} = \begin{cases} a_i + u_o & i = 1, 2, \cdots, m \\ u_o & i = m+1, \cdots, .m+n \end{cases} \tag{3.2}$$

$$\sum_{i=1}^{m+n} x_{ij} = \begin{cases} u_o & j = 1, 2, \cdots, m \\ d_j + u_o & j = m+1, \cdots, m+n \end{cases} \tag{3.3}$$

$$x_{ij} \geq 0 \tag{3.4}$$

Here the third term of the right hand side viz. $\sum_{i=1}^{m+n} c_i\, u_o$ can be adjusted in the second term as $u_i = u_o - x_{ii}$, and is treated as constant.

4. THE EQUIVALENT DETERMINISTIC PROBLEM

Let the demand d_j's at various destinations be independent random variables and the probability distribution of d_j $(j = 1, 2, \cdots, n)$ be in increasing order as follows:

Demand d_j	$d_{1j} <$	$d_{2j} <$	$\cdots$	$d_{H_j j}$
$p(d_j = d_{hj}) = p_{hj}$	p_{1j}	p_{2j}	$\cdots$	$p_{H_j j}$
$p(d_j \geq d_{hj}) = \pi_{hj}$	$\pi_{1j} = \sum_{h=1}^{H_j} p_{hj}$	$\pi_{2j} = \sum_{h=2}^{H_j} p_{hj}$	$\cdots$	$\pi_{H_j j} = p_{H_j j}$

To determine the function $\phi_j(s_j, y_j)$ note that y_j, the net quantity shipped to sink j, can be any amount between the lowest value d_{1j} and the highest value $d_{H_j j}$ in the probability distribution of the demand d_j $(j = 1, 2, \cdots, n)$.

If $0 \le y_j \le d_{1j}$, then each of the y_j units shall be absorbed with probability $\pi_{1j} (=1)$.

Hence, the expected revenue is $= s_j \pi_{1j} y_j$.

If $d_{1j} \le y_j \le d_{2j}$, then each unit upto d_{1j} shall be absorbed with probability π_{1j} and each of the additional units $(y_j - d_{1j})$ shall be absorbed with probability π_{2j}.

Hence, the expected revenue is $= s_j \pi_{1j} d_{1j} + s_j \pi_{2j} (y_j - d_{1j})$.

So, in general, if $d_{hj} \le y_j \le d_{h+1j}$, then the expected revenue is

$$s_j \{\pi_{1j} d_{1j} + \pi_{2j} (d_{2j} - d_{1j}) + \cdots + \pi_{hj} (d_{hj} - d_{h-1j}) + \pi_{h+1j} (y_j - d_{hj})\}$$

.

Let us now break y_j into incremental units y_{hj} $(h = 1, 2, \cdots, H_j)$ as:

$$y_j = y_{1j} + y_{2j} + \cdots + y_{hj} + \cdots + y_{H_j j}, \tag{4.1}$$

where
$$\left.\begin{array}{ll} 0 \le y_{1j} \le d_{1j} & = F_{1j} \\ 0 \le y_{2j} \le d_{2j} - d_{1j} & = F_{2j} \\ \vdots \quad \vdots \quad \vdots & \vdots \\ 0 \le y_{H_j j} \le d_{H_j j} - d_{H_j - 1, j} & = F_{H_j j} \end{array}\right\} \tag{4.2}$$

Relation (4.1) makes physical sense only if there exists some $h = h_j$ (say) such that all intervals below the h_j –th interval are filled to capacity and all intervals above it are empty i.e.

$$\left.\begin{array}{ll} y_{hj} = F_{hj} & (h = 1, 2, \cdots, h_j - 1) \\ y_{hj} \le F_{hj} & (h = h_j) \\ y_{hj} = 0 & (h = h_j + 1, \cdots, H_j) \end{array}\right\} \tag{4.3}$$

Assuming for the time being that the conditions (4.3) hold, the total expected revenue from sink j is:

$$\phi_j (s_j, y_j) = \sum_{h=1}^{H_j} s_j \pi_{hj} \, y_{hj} \tag{4.4}$$

Substituting the value of $\phi_j (s_j, y_j)$ from (4.4) in (3.1) and treating both x_{ij} and y_{hj} as decision variables, the deterministic equivalent to Problem P_3, is:

Problem P_4:

$$\text{Min} \quad Z = \frac{\sum_{i=1}^{m+n}\sum_{j=1}^{m+n} \rho_{ij} x_{ij} + \sum_{j=1}^{m+n} \beta_{hj} y_{hj}}{\sum_{i=1}^{m+n}\sum_{j=1}^{m+n} c_{ij} x_{ij}} = \frac{Z_1}{Z_2}, \tag{4.5}$$

where $(\beta_{hj} - s_j \pi_{hj})$.

$$\text{Subject to} \quad \sum_{j=1}^{m+n} x_{ij} = \begin{cases} a_i + u_o & i = 1, 2, \cdots, m \\ u_o & i = m+1, \cdots, m+n \end{cases} \tag{4.6}$$

$$\sum_{i=1}^{m+n} x_{ij} = u_0, \qquad j = 1, 2, \cdots, m \tag{4.7}$$

$$\sum_{i=1}^{m+n} x_{ij} - \sum_{h=1}^{H_j} y_{hj} = u_0, \qquad j = m+1, \cdots, m+n \tag{4.8}$$

$$x_{ij},\ y_{hj} \geq 0 \quad (\forall i, j, h) \tag{4.9}$$

$$y_{hj} \leq F_{hi} \ (\forall h, j) \tag{4.10}$$

and subject to the additional stipulation that the constraints (4.3) are also satisfied.

Fortunately, it turns out that (4.3) do not restrict our choice of optimum solution in any way. This can be handled by the theorem as given by Javaid *et al.* (1998).

5. PRELIMINARIES TO THE SOLUTION OF PROBLEM P_4

i) It is assumed that the set of all feasible solutions of Problem P_4 is regular (i.e. non- empty and bounded) and that the denominator of the objective function is positive for all feasible solution, Bela Martos (1968) and Cooper (1962).

ii) Since the deterministic Problem P_4 is a transportation type fractional linear programming problem, its global minimum exists at a basic feasible solution of its constraints, Javaid *et al.* (1998).

iii) A global minima to the problem exist at a basic feasible solution to the capacitated system.

iv) As none of the equations (4.6) and (4.7) is redundant, a basic feasible solution to the original system shall contain $(m+n)$ basic variables. We shall, hereinafter, call the constraints (4.5) through (4.9) as the original system and the constraints (4.5) through (4.10) as the capacitated system.

v) As none of the constraints in the original system is redundant, a basic feasible solution to the original system shall contain $2(m+n)$ basic variables. For the capacitated system also, a basic feasible solution shall contain $2(m+n)$ basic variables and the same may be found by working on the original system provided that some of the non basic variables are allowed to take their upper bound values, Dantzig (1963).

The special structure of Problem P_4, permits us to arrange it into an array as shown below, Garwin (1963).

Table 1:

x_{11} ρ_{11} 0	$\cdots$	x_{1m} ρ_{1m} c_{1m}	x_{1m+1} ρ_{1m+1} c_{1m+1}	$\cdots$	x_{1m+n} ρ_{1m+n} c_{1m+n}	a_1+t_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{m1} ρ_{m1} c_{m1}	$\cdots$	x_{mm} ρ_{mm} c_{mm}	x_{mm+1} ρ_{mm+1} c_{mm+1}	$\cdots$	x_{mm+n} ρ_{mm+n} c_{mm+n}	a_m+t_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{m+n1} ρ_{m+n1} c_{m+n1}	$\cdots$	x_{m+nm} ρ_{m+nm} c_{m+nm}	x_{m+nm+1} ρ_{m+nm+1} c_{m+nm+1}	$\cdots$	x_{m+nm+n} ρ_{m+nm+n} c_{m+nm+n}	a_m+t_0
t_0	$\cdots$	t_0	y_{1m+1} F_{1m+1} β_{1m+1} α_{1m+1}	$\cdots$	y_{1m+n} F_{1m+n} β_{1m+n} α_{1m+n}	
			$\vdots$	$\vdots$	$\vdots$	$\vdots$
			y_{Hm+1} F_{Hm+1} β_{Hm+1} α_{Hm+1}	$\cdots$	y_{Hm+n} F_{Hm+n} β_{Hm+n} α_{Hm+n}	
			t_0		t_0	

In the above table, there are $(m+n)$ rows in columns $j=1,2,\cdots,m$ and $(m+n+H)$ rows in columns $j=m+1,\cdots,m+n$. Here $H=\max.H_j$, so that there shall be some empty boxes near the bottom of the table in columns $j=m+1,\cdots,m+n$. These empty boxes shall be crossed out.

Absence of the row totals for y_{hj}'s in the table indicates that there are no row equations for y_{hj} variables. Besides, to obtain the column equations (4.8), each y_{hj} has to be multiplied by (-1). We have omitted (-1) from y_{hj} boxes for convenience.

6. INITIAL BASIC FEASIBLE SOLUTION AND OPTIMALITY CRITERIA

To start with, we fix the demands $d_j's$ approximately equal to their expected values such that $\sum_{j=m+1}^{m+n} d_j = \sum_{i=1}^{m} a_i$ and also such that for all j except $j = j^*$, each d_j falls at the upper end of one of the intervals y_{hj} into which d_j has been divided, i.e., $d_j = \sum_{h=1}^{h'_j} F_{hj}$ for some $h'_j \le H_i$ and for all j except $j = j^*$ (the d_j can always be so chosen that it is done).

With these fixed demands the upper portion of the Table 1 resembles a $(m+n)\times(m+n)$ standard transportation problem for which an initial basic feasible solution with $\{2(m+n)-1\}$ basic variables is obtained by any of the several available methods. Now, in each of the columns $j = m+1,\cdots,m+n$, the values of the non basic $y_{hj}'s$ are entered at their upper bounds in turn $h = 1,2,\cdots$ until we have entered enough non basic $y_{hj}'s$ so that their sum over h is equal to d_j. Obviously, we shall never have to enter y_{hj} below its upper bound except in column $j = j^*$, where the last nonzero entry will be $y_{hj^*} \le F_{hj^*}$. This last entry and the $\{2(m+n)-1\}$ basic $x_{ij}'s$ found earlier constitute the required initial basic feasible solution with $2(m+n)$ basic variables. In case the last non zero entry in column j^* is also at its upper bound, then we take the last y_{hj} entry of any column as our $2(m+n)-$ th basic variable.

Let the simplex multipliers corresponding to the objective function Z_1 (Problem P$_4$) be u_i and v_j $(\forall\ i, j = 1,2,\cdots,m+n)$ and Z_2 be μ_i and υ_i $(\forall\ i, j = 1,2,\cdots,m+n)$

These are determined by solving the following equations.

$$\left.\begin{aligned} \rho_{ij} + u_i + v_j &= 0 \quad \text{for basic } x_{ij} \\ \beta_{hj} - v_j &= 0 \quad \text{for basic } y_{hj} \end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned} c_{ij} + \mu_i + \upsilon_j &= 0 \quad \text{for basic } x_{ij} \\ -v_j &= 0 \quad \text{for basic } y_{hj} \end{aligned}\right\} \tag{6.2}$$

Each of the system (6.1) and (6.2) have $2(m+n)$ linear equations in as many unknowns u_i, v_j, μ_i and υ_j and can be easily solved. Let the relative cost coefficients corresponding to the variables x_{ij} and y_{hj} be ρ'_{ij} and β'_{hj} for Z_1 and C'_{ij} and α'_{hj} for Z_2.

These are determined by solving the following equations

$$\left.\begin{aligned} \rho'_{ij} &= \rho_{ij} + u_i + v_j \quad && \text{for non basic } x_{ij} \\ \beta'_{hj} &= \beta_{hj} - v_j && \text{for non basic } y_{hj} \end{aligned}\right\} \tag{6.3}$$

$$\left.\begin{aligned} C'_{ij} &= c_{ij} + \mu_i + \upsilon_j \quad && \text{for non basic } x_{ij} \\ \alpha'_{hj} &= -v_j && \text{for non basic } y_{hj} \end{aligned}\right\} \tag{6.4}$$

The relative cost coefficients for basic variables and the values of the non basic x_{ij}'s are zero. As regards the values of non basic y_{hj}'s - some are zero and others at upper bounds.

It can be easily shown that for a given basic feasible solution (x_{ij}, y_{hj}) of the Problem P_3, the value of the objective function Z is

$$Z = \frac{\sum_{i=1}^{m+n}\sum_{j=1}^{m+n}\rho'_{ij}x_{ij} + \sum_{j=1}^{m+n}\sum_{h=1}^{H_j}\beta'_{hj}y_{hj} - \left\{\sum_{i=1}^{m}u_i\,(a_i + u_0) + \sum_{j=1}^{m+n}v_j u_0\right\}}{\sum_{i=1}^{m+n}\sum_{j=1}^{m+n}C'_{ij}x_{ij} + \sum_{j=1}^{m+n}\sum_{h=1}^{H_j}\alpha'_{hj}y_{hj} - \left\{\sum_{i=1}^{m}\mu_i\,(a_i + u_0) + \sum_{j=1}^{m+n}\upsilon_j u_0\right\}} = \frac{Z_1}{Z_2}$$

(6.5)

Here, $C'_{ij} = 0$ for all basic x_{ij} and also the values of the non basic x_{ij} are zero. So, the first term on the right hand side of (6.3) vanishes. Similarly $R'_{hj} = 0$ for the basic y_{hj}, but as regards the values of non basic y_{hj}'s - some are zero and the others are at their upper bounds. Hence,

$$Z = \frac{\sum_{i=1}^{m+n}{}^{*}\sum_{h=1}^{H_j}{}^{*}\beta'_{hj}F_{hj} - \left\{\sum_{i=1}^{m}u_i\,(a_i + u_0) + \sum_{j=1}^{m+n}v_j u_0\right\}}{\sum_{i=1}^{m+n}{}^{*}\sum_{j=1}^{H_j}{}^{*}\alpha'_{hj}F_{hj} - \left\{\sum_{i=1}^{m}\mu_i\,(a_i + u_0) + \sum_{j=1}^{m+n}\upsilon_j u_0\right\}} = \frac{Z_1}{Z_2}, \tag{6.6}$$

where $\sum^{*}$ indicates the sum over those non basic y_{hj} which are at their upper bounds. Now if the value of any one of the non basic variables x_{st} or y_{rt} is changed to

$\hat{x}_{st} = (x_{st} + \theta)$ or $\hat{y}_{rt} = (y_{rt} \pm \theta)$

with the other non basic variables remaining unaltered and the basic variables adjusted to maintain feasibility of the solution, then the improved value of Z shall be

$$\hat{Z} = \frac{Z_1 + \theta.\rho'_{st}}{Z_2 + \theta.C'_{st}} \qquad \text{or} \qquad \hat{Z} = \frac{Z_1 + \theta.\beta'_{rt}}{Z_2 + \theta.\alpha'_{rt}}$$

as the case may be. It is important to note that we take plus sign if $y_{rt} = 0$ and minus sign if $y_{rt} = F_{rt}$.

The objective function will improve iff $\hat{Z} - Z < 0$, i.e.

$$\frac{Z_1 + \theta.\rho'_{st}}{Z_2 + \theta.C'_{st}} - \frac{Z_1}{Z_2} < 0 \quad \text{or} \quad \pm \frac{Z_1 + \theta.\beta'_{rt}}{Z_2 + \theta.\alpha'_{rt}} - \frac{Z_1}{Z_2} < 0$$

i.e., $\theta(p'_{st}Z_2 - C'_{st}Z_1) < 0$ or $\pm\theta(\beta'_{rt}Z_2 - \alpha'_{rt}Z_1) < 0$.

Since, in the non degenerate case $\theta > 0$ and in degenerate case $\theta = 0$,

$\Rightarrow \hat{Z} = Z$.

Defining $\begin{cases} B_{ij} = p'_{ij}Z_2 - C'_{ij}Z_1 \\ \overline{B}_{hj} = \beta'_{hj}Z_2 - \alpha'_{hj}Z_1 \end{cases}$

Thus, the current solution is optimum iff

$$\left.\begin{array}{ll} B_{ij} \geq 0 & (\forall \text{ non basic } x_{ij}) \\ \overline{B}_{hj} \geq 0 & (\forall \text{ non basic } y_{ij} \text{ at zero level}) \\ \overline{B}_{hj} \leq 0 & (\forall \text{ non basic } y_{hj} \text{ at upper bound}) \end{array}\right\} \qquad (6.7)$$

If any of the optimality criteria (6.7) is violated, the current solution can be improved. The non basic variable which violates (6.7) most severely is selected to enter the basis. The values of the new basic variables are found by applying the usual θ-adjustments. It should, however, be kept in mind that the coefficient of each y_{hj} in column equations (4.8) is (-1). The variable to leave the basis is the one that becomes either zero or equal to its upper bound. If two or more basic variables reach zero or their upper bounds simultaneously then only one of them becomes non basic. Should it happen that the entering variable itself attains upper or lower bound (zero) without simultaneously making any of the basic variables zero or equal to its upper bounds, the set of basic variables remains unaltered; only their values are changed to allow the so-called entering variable to be fixed at its upper or lower bound.

Finiteness

The process is bound to terminate with a finite number of iterations as it involves movement from one basic feasible solution to another basic feasible solution, which is finite in number.

7. ALGORITHM OF THE DETERMINISTIC PROBLEM

The step-by-step computational algorithm for determining the optimum solution is given as follows:

Step 1- First of all calculates initial/improved basic feasible solution and records them in a working table.

Step 2- Then obtain the values of simplex multipliers $(u_i, v_j, \mu_i$ and $v_j)$ and relative cost coefficients by given equations from equations (6.1), (6.2), (6.3) and (6.4) and record them in current working table.

Step 3- Calculate the value of the objective function Z by the equation (4.5).

Step 4- Then for the non-basic variables, calculate B_{ij} and $\overline{B}_{hj}$ and test whether the solution is optimum or not. If yes, the process terminates and if not, proceed to find the B_{ij} (or $\hat{\beta}_{hj}$) which violates the optimality criteria (6.7), most severely.

Step 5- Find the entering variable as the one who's corresponding B_{ij} (or $\hat{\beta}_{hj}$) violates the optimality criteria most severely.

Step 6- Apply $\theta -$adjustments and determine the outgoing variable (if any) and find the maximum value θ.

Step 7- Go to step 1.

8. NUMERICAL EXAMPLE

Consider the transportation problem from source to sink with a_i, c_{ij} and s_j as given in Table $(N-1)$ and the probability distribution of d_j in able $(N-5)$. The values of R_{hj} and F_{hj} are calculated in Table $(N-5)$. The various costs associated with transshipment are given in Table $(N-2)$ and Table $(N-3)$. Also the initial basic feasible solution of the transportation problem by North-west corner rule also given in Table $(N-4)$.

Working tables for determining optimal solution are given in Table $(N-6)$, in which the entries in x_{ij} and y_{hj} boxes are as follows:

x_{ij}	B_{ij}
ρ'_{ij}	C'_{ij}

y_{hj}	$\overline{B}_{hj}$
β'_{hj}	α'_{hj}

Here, the presence of smaller box in the y_{hj} box indicates a non-basic variable at its upper bound shipping.

	A		*B*		a_i
I	0	5	2	3	10
II	0	4	1	2	5
III	1	7	0	1	6
s_j	10		5		

Table (N-1): Transportation from source to sink

	I		II		III	
I	0	0	0	5	4	6
II	2	4	0	0	3	2
III	4	6	3	2	0	0

Table (N-2): Transshipment from source to source

	A		*B*	
A	0	0	2	3
B	2	3	0	0

Table (N-3): Transshipment from sink to sink

	A		*B*		a_i
I	9				
	0	5	2	3	9
II	3		2		
	0	4	1	2	5
III			6		
	1	7	0	1	6
d_j	12		9		

Table (N-4): *IBF* Solution by North-West Corner Rule

j	d_j	p_{hj}	π_{hj}	$R_{hj} = s_j \pi_{hj}$	F_{hj}
1	9	0.2	1.0	-10	9
	12	0.6	0.8	-7	3
	16	0.2	0.2	-2	5
2	7	0.2	1.0	-5	7
	10	0.8	0.8	-4	3

Table (N-5): Assumed distribution of d_j

Iteration-1

Step-1:- In order to obtain initial basic feasible solution we fix the demands at $b_1 = 12$ and $b_2 = 9$, and then we determine a starting basic feasible solution to the standard transportation problem by the North-West corner Rule. Then a standard transshipment problem is formed with the initial basic solution as:

$$x_{11} = 21, \quad x_{14} = 10, \quad x_{22} = 21,$$

$$x_{24} = 2, \quad x_{25} = 3, \quad x_{33} = 21,$$

$$x_{35} = 6, \quad x_{44} = 21, \quad x_{55} = 21.$$

To obtain the *IBFS* to the deterministic equivalent transportation problem, we assign y_{hj} entries at their upper bounds (as far as possible) so that the column equations are satisfied.

We get,

$$y_{11} = 9, \quad y_{21} = 3, \quad y_{12} = 7$$

and

$$y_{22} = 2 \ (< R_{22}).$$

This provides the required initial basic feasible solution with x_{11}, x_{21}, x_{22}, x_{32} and y_{22} as the basic variables.

Step-2:- The simplex multipliers (u_i, v_j, μ_i and v_j) and the relative cost coefficients are determined.

Step 3:- The value of Z, by using equation 6.6, is found to be -2.157.

					a_i
[20] 0	2 4	4 6	[9] 0 5	2 3	31
2 4	[20] 0 0	3 2	[3] 0 4	[3] 1 2	26
4 6	3 2	[20] 0 0	1 7	[6] 0 1	27
0 5	4 1	1 7	[20] 0 0	5 4	21
2 3	1 2	0 1	5 4	[20] 0 0	21
21	21	21			
			[9] −5 −10	[6] −1 −5	
			[3] −3 −7	[2] −4	
			−2		
			21	21	

Table (N-6): Deterministic version of Problem P_4

After the ***Third iteration*** the optimal solution has been attained as:

$$Z_{opt} = -2.16,$$

$$x_{11} = 21,\ x_{14} = 10,\ x_{22} = 21,\ x_{24} = 1,\ x_{25} = 4,\ x_{33} = 21,\ x_{35} = 6,\ x_{44} = 21,\ x_{55} = 21.$$

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REFERENCES

Bela Martos (1968): On varying all the parameters in a linear fractional functional programming problem. *Metrika*, **13**, 196 – 205.

Charnes, A. and Cooper, W.W. (1962): Programming with linear fractional functional. *Naval Res. Logist. Quart.*, **9**, 181-186.

Dantzig G.B. (1963): *Linear Programming and Extensions*. Princeton University Press.

Ferguson, A.R. and Dantzig G.B. (1956): The allocation of aircraft to routes-an example of linear programming under uncertain demand. *Management Sci.*, **3**, 45-73.

Garvin W.W. (1963): *Introduction to Linear Programming*. McGraw Hill Book Company, New York.

Javaid, S., Gupta S.N. and M.M. Khalid (2000-01): Stochastic fractional transshipment problem for maximizing profitability. *J. Decis. Math. Sci.*, **5-6**, 1-10.

Javaid, S., Gupta S.N. and M.M. Khalid (1998): An uncapacitated stochastic fractional transshipment problem for maximizing marketing efficiency. *Int. J. Management Sci.*, **14**, 257-263.

Orden, A. (1956): The transshipment problem. *Management Sci.*, **2**, 276-285.

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