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SOME NON-ADDITIVE GENERALIZED MEASURES OF 'USEFUL' INFORAMTION AND J-DIVERGENCE

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ABSTRACT

In the present paper an axiomatic characterization of non-additive measures of 'useful' information associated with a pair of probability distributions of a sample space having utility distribution corresponding to the same number of elements in both probability distributions has been studied. The quantity so obtained under additional suitable postulates leads to the generalized measures of 'useful' relative information, information improvement and J-divergence. Particular cases and important properties of the measures so obtained have also been studied.

1. INTRODUCTION

Let (Ω, A, P) be the probability space, where Ω is a set of all possible outcomes in an experiment, A is $\sigma-$algebra of all subsets of Ω i.e. a set of all possible events and P is the probability measure such that $P(A_k) = p_k$ for each $A_k \in A$.

Let us consider a generalized probability distribution

$P = \{(p_1, p_2, \cdots, p_n)\,;\ 0 < p_i \le 1$ for each i and $\sum_{i=1}^{n} p_i \le 1\}$ together with utility distribution $U = (u_1, u_2, \cdots, u_n)$; where $u_k > 0$ is the utility or importance of an event $A_k \in A$ and is independent of its probability of occurrence p_k.

Let

$$I_n : \Delta_n \times R_+^n \to R_+, \qquad n \ge 2$$

where $\Delta_n = \{(p_1, p_2, \cdots, p_n)\,;\ 0 < p_i \le 1 \text{ for each } i \text{ and } \sum_{i=1}^{n} p_i \le 1\}$

$$R_+ = (0, \infty) \text{ and } R_+^n = \{(u_1, u_2, \cdots, u_n)\,;\ u_i > 0 \text{ for every } i\}$$

Then Belis and Guiasu (1968) quantitative-qualitative measure is defined

$$I_n\{(p_1, p_2, \cdots, p_n); (u_1, u_2, \cdots, u_n)\} = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} p_i} \tag{1.1}$$

which was called 'useful' information measure by Longo (1972).

Measure (1.1) satisfies additivity of the following type:

$$I_{nm}(P * Q;\ U * V) \equiv \overline{V}\, I_n\,(P;U) + \overline{U}\, I_m\,(Q;V) \tag{1.2}$$

where $P = \left\{ (p_1, p_2, \cdots, p_n),\ p_i > 0,\ \sum_{i=1}^{n} p_i \le 1 \right\}$

$$Q = \left\{ (q_1, q_2, \cdots, q_m),\ q_j > 0,\ \sum_{j=1}^{m} q_j \le 1 \right\}$$

and

$$P * Q = (p_1 q_1, p_1 q_2, \cdots, p_1 q_m, \cdots, p_n q_1, p_n q_2, \cdots, p_n q_m)$$

$$U * V = (u_1 v_1, u_1 v_2, \cdots, u_1 v_m, \cdots, .u_n v_1, u_n v_2, \cdots, u_n v_m)$$

$$\overline{U} = \frac{\sum_{i=1}^{n} p_i u_i}{\sum_{i=1}^{n} p_i} \quad \text{and} \quad \overline{V} = \frac{\sum_{j=1}^{m} q_j v_j}{\sum_{j=1}^{m} q_j}$$

Further by considering posterior probability distribution $P \in \Delta_n$ on the basis of an experiment where predicted probability distribution is $Q \in \Delta_n$ and having the utility distribution $U \in R_+^n$, the following measure was defined and characterized by Taneja and Tuteja (1984, 1985) for complete probability distributions:

$$I_n(P/Q;\ U) = \sum_{i=1}^{n} u_i p_i \log(p_i / q_i) \tag{1.3}$$

The measure satisfies the additivity of the type (1.2) and is called 'useful' relative information or directed divergence measure with preference.

The following non-additive 'useful' information of degree β was first introduced and characterized by Sharma *et al.* (1978):

$$I^{\beta}(P;U)=\frac{\sum_{i=1}^{n}u_i p_i(p^{\beta-1}-1)}{(2^{1-\beta}-1)\sum_{i=1}^{n}p_i},\quad \beta>0 \text{ and } \beta\neq 1 \tag{1.4}$$

Corresponding to (1.4) Taneja (1985) characterized and studied the generalized measure of 'useful' relative information of degree β given by

$$I^{\beta}(P/Q;U)=\frac{\sum_{i=1}^{n}u_i p_i[(p_i/q_i)^{\beta-1}-1]}{(2^{\beta-1}-1)\sum_{i=1}^{n}p_i},\quad \beta>0 \text{ and } \beta\neq 1 \tag{1.5}$$

In case $\beta\to 1$, the measures (1.4) and (1.5) reduce to (1.1) and (1.3) respectively. Further, if utilities are ignored these reduce respectively to Shannon's entropy and Kulback's measure of relative information for generalized probability distributions.

Hooda and Tuteja (1981) further generalized the measure given by (1.4) and obtained the following measure:

$$I_{\alpha}^{\beta}(P;U)=\frac{\sum_{i=1}^{n}u_i^{\alpha} p_i^{\alpha}(p_i^{\beta-\alpha}-1)}{(2^{1-\beta}-2^{1-\alpha})\sum_{i=1}^{n}p_i} \tag{1.6}$$

where α and β are arbitrary real constants satisfying either $\alpha.>1$, $0<\beta\le 1$ or $\beta>1$, $0<\alpha\le 1$. This measure was called as the generalized measure of type α and degree β. It reduces to (1.4) when $\alpha=1$.

Let $R\in\Delta_n$ be the revised probability distribution of a predicted probability distribution $Q\in\Delta_n$, where P is the actually realized probability distribution in an experiment having utility distribution $U=(u_1,u_2,\cdots,u_n)$ then the 'useful' relative information from P to Q is given by

$$I(P/Q;U)=\frac{\sum_{i=1}^{n}u_i p_i\log(p_i/q_i)}{\sum_{i=1}^{n}p_i} \tag{1.7}$$

and the 'useful' relative information from P to Q is given by

$$I(P/R;U)=\frac{\sum_{i=1}^{n}u_i p_i \log(p_i/r_i)}{\sum_{i=1}^{n}p_i} \tag{1.8}$$

The 'useful' information improvement measure is defined as

$$I(P:Q:R;U)=I(P/Q;U)-I(P/R;U) = \frac{\sum_{i=1}^{n}u_i p_i \log(r_i/q_i)}{\sum_{i=1}^{n}p_i} \tag{1.9}$$

which reduces to Singh and Bhardwaj's (1991) measure of 'useful' information improvement in case $\sum_{i=1}^{n}p_i=1$.

In the present communication, some non-additive generalized measures of 'useful' relative information, information improvement and $J-$divergence containing the parameters α and β have been characterized axiomatically by a general method. Their particular cases and important properties have also been studied.

2. CHARACTERIZATION OF NON-ADDITIVE 'USEFUL' INFORMATION

Let (Ω, A, P) be the probability space as defined in section 1. Let P and Q be respectively posterior and prior generalized probability distributions and U be a utility distribution of some goal oriented experiment defined on A, where U in general is independent of P and Q.

Let

$$I_n^*:\Delta_n^*\times R_+^n\to R_+, \qquad n\geq 2$$

where

$$\Delta_n^*=\left\{(p_1,p_2,\cdots,p_n)/(q_1,q_2,\cdots,q_n);0<p_i,q_i\leq 1, \sum_{i=1}^{n}p_i\leq 1, \sum_{i=1}^{n}q_i\leq 1\right\}$$

$$R_+^n=\{(u_1,u_2,\cdots,u_n);u_i>0 \ \text{ for every } i\}$$

and

$R_+=(0,\infty)$ satisfies the following postulates:

Postulate 2.1:

$I^*(\{1\}/\{q\};\{1\})$, $I^*(\{p\}/\{1\};\{1\})$ and $I^*(\{1\}/\{1\};\{u\})$ are continuous functions of $p,q\in(0,1]$ and $u\in(0,\infty)$.

Postulate 2.2:

$$I_n^*[\{p_1 x, p_2 x,\cdots,p_n x\}/\{q_1 y, q_2 y,\cdots,q_n y\};\{u_1 z, u_2 z,\cdots,u_n z\}]$$

$$= z^\beta I_n^*[\{p_1,p_2,\cdots,p_n\}/\{q_1,q_2,q_n\};\{u_1,u_2,\cdots,u_n\}]$$

$$+\sum_{i=1}^n u_i^\beta I^*[\{x\}/\{y\};\{z\}]$$

$$+cI_n^*[\{p_1,p_2,\cdots,p_n\}/\{q_1,q_2,\cdots,q_n\};\{u_1,u_2,\cdots,u_n\}]$$

$$I^*[\{x\}/\{y\};\{z\}]$$

where $c\neq 0$ and $\beta>0$ are some arbitrary constants.

Now we prove the following lemma:

Lemma 2.1: The measure $I^*(\{p\}/\{q\};\{u\})$ of 'useful' information for $p,q\in(0,1]$, $u>0$ and satisfying postulates (2.1) and (2.2) is given by

$$I^*(\{p\}/\{q\};\{u\})=\frac{u^\beta(p^\delta q^\lambda u^\mu-1)}{2^{\alpha-1}-1},\quad \alpha\neq 1,\quad \beta\geq 0 \tag{2.1}$$

And $\delta,\lambda,\ \mu$ are arbitrary constants

Proof:

Let

$$I^*(\{p\}/\{q\};\{u\})=f(p/q\ ;u) \tag{2.2}$$

Then postulate (2.2) for $n=1$ gives

$$f(px/qy;\ uz)=z^\beta f(p/q;\ u)+u^\beta f(x/y;z)+c\,f(p/q\ ;\ u)\,f(x/y;z) \tag{2.3}$$

Setting $f(p/q;u)=u^\beta\phi(p/q;u)$ in (2.3), we have

$$\phi(px/qy;uz)=\phi(p/q;u)+\phi(x/y\ ;z)+c\,\phi(p/q;u)\phi(x/y;z) \tag{2.4}$$

and

$$\phi(p/q;u)=\phi(p/q;1)+\phi(1/1;u)+c\,\phi(p/q;1)\ \phi(1/1;u) \tag{2.5}$$

Further

$$\phi(p/q;1)=\phi(p/1;1)+\phi(1/q;1)+c\,\phi(p/1;1)\,\phi(1/q;1) \tag{2.6}$$

Again setting $p=q=1$, $x=y=1$ in (2.4), we have

$$\phi(1/1;uz)=\phi(1/1;u)+\phi(1/1;z)+c\,\phi(1/1;u)\,\phi(1/1;z) \tag{2.7}$$

Equation (2.7) can be written as

$$1+c\phi(1/1;uz)=[1+c\phi(1/1;u][1+c\phi(1/1;z)] \tag{2.8}$$

Setting $1+c\phi(1/1;u)=H(u)$ in (2.8) we get

$$H(uz)=H(u)\,H(z) \tag{2.9}$$

By postulate (2.1), $\phi(1/1;u)$ is a continuous function of u, hence $H(u)$ is also a continuous function of *u*. $H(u)\equiv 0$ is also a solution, but it is insignificant. Thus we consider only non-zero continuous solution of (2.9), which is of the form $H(u)=u^{\mu}$, μ being some arbitrary constant. [Refer Aczel (1966), P41].

Therefore

$$\phi(1/1;u)=\frac{u^{\mu}-1}{c} \tag{2.10}$$

where μ and $c\neq 0$ are arbitrary constants.

Similarly

$$\phi(1/q;1)=\frac{q^{\lambda}-1}{c} \tag{2.11}$$

and

$$\phi(p/1;1)=\frac{p^{\delta}-1}{c} \tag{2.12}$$

where λ, δ and $c\neq 0$ are arbitrary constants.

Putting (2.11) and (2.12) in (2.6), we get

$$\phi(p/q;1)=\frac{p^{\delta}q^{\lambda}-1}{c} \tag{2.13}$$

Putting (2.13) and (2.10) in (2.5), we have

$$\phi(p/q;u)=\frac{p^{\delta}q^{\lambda}u^{\mu}-1}{c} \tag{2.14}$$

Finally from $f\,(p\,/\,q;\,u) = u^{\beta}\phi(p\,/\,q;u)$, we get

$$f(p\,/\,q;u) = \frac{u^{\beta}(p^{\delta}q^{\lambda}u^{\mu}-1)}{c} \tag{2.15}$$

where β, λ, δ, μ and $c \neq 0$ are arbitrary constants. In particular we can take

$c = 2^{\alpha-1}-1$, $\alpha \neq 1$ in (2.15), which give (2.1). Next we characterize the non-additive measure of 'useful' relative information in the following theorem.

Theorem 2.1: The non-additive measure of 'useful' information $I\,(P\,/\,Q;\,U)$ satisfying postulates 2.1, 2.2 and

Postulate 2.3:

$$I^{*}\{1\}/\{1/2\};\{1\})=1$$

Postulate 2.4: In particular,

$$I^{*}(P\,/\,P;U)=0$$

$$I^{*}(\{1\}/\{1\};\{u\}) = I^{*}(\{1/2\}/\{1/2\};\{1\})=0$$

Postulate 2.5:

$$I_n^{*}(\{p_1,p_2,\cdots,p_n\}/\{q_1,q_2,\cdots,q_n\};\{u_1,u_2,\cdots,u_n\})$$

$$= \frac{\sum_{k=1}^{n} w_{\beta}(p_k)I^{*}(\{p_k\}/\{q_k\};\{u_k\})}{\sum_{k=1}^{n} w_{\beta}(p_k)} \tag{2.16}$$

where

$$\beta(>0)\text{, such that } \sum_{k=1}^{\mathrm{n}} w_{\beta}(p_k) = p_1^{\beta}+p_2^{\beta}+\cdots+p_k^{\beta} \leq 1$$

is given by

$$I^{*}(P\,/\,Q;U) = \frac{\sum_{i=1}^{n} u_i^{\beta}p_i^{\beta}[(p_i\,/\,q_i)^{\alpha-1}-1]}{(2^{\alpha-1}-1)\sum_{i=1}^{n} p_i^{\beta}} \tag{2.17}$$

where $u_i > 0$, $0 < p_i$, $q_i \leq 1$ for each i, $\beta > 0$ and $\alpha \neq 1$

Proof: By postulates 2.3 and 2.4, (2.15) gives

$$\lambda = 1-\alpha,\ \mu = 0 \text{ and } \delta = \alpha - 1 \tag{2.18}$$

Substituting the values from (2.18) in (2.15), we get

$$f(p/q;u) = \frac{u^{\beta}(p^{\alpha-1}q^{1-\alpha}-1)}{2^{\alpha-1}-1} \tag{2.19}$$

By using postulate 2.5 in (219), we have

$$I_n^*(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, .q_n\}; \{u_1, u_2, \cdots, u_n\})$$

$$= \frac{\sum_{i=1}^{n} u_i^{\beta} p_i^{\beta}[(p_i/q_i)^{\alpha-1}-1]}{(2^{\alpha-1}-1)\sum_{i=1}^{n} p_i^{\beta}} = I_{\alpha}^{\beta}(P/Q;U),\ \alpha,\ \beta > 0 \text{ and } \alpha \neq 1 \tag{2.20}$$

We may call (2.20) as the non-additive generalized measure of 'useful' relative information of order α and degree β.

Particular Cases:

Case 2.1) When $\beta = 1$ (2.20) reduces to

$$I_{\alpha}(P/Q;U) = \frac{\sum_{i=1}^{n} u_i\ p_i[(p_i/q_i)^{\alpha-1}-1]}{(2^{\alpha-1}-1)\sum_{i=1}^{n} p_i}, \quad \alpha(\neq 1) > 0,\ \sum_{i=1}^{n} p_i \leq 1 \tag{2.21}$$

which is the non-additive 'useful' relative information of order α studied by Taneja (1985). In case $\sum_{i=1}^{n} p_i = 1$, then (2.21) reduces to

$$I_{\alpha}(P/Q;U) = \frac{\sum_{i=1}^{n} u_i\ p_i(p_i/q_i)^{\alpha-1}-1}{(2^{\alpha-1}-1)}, \alpha(\neq 1) > 0 \tag{2.22}$$

which is non-additive generalized measure of 'useful' relative information characterized by Hooda (1984). Further

$$\lim_{\alpha \to 1} I_\alpha (P/Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i / q_i)}{\sum_{i=1}^{n} p_i} \tag{2.23}$$

which is 'useful' relative information defined and characterized by Hooda (1983).

Case 2.2) If utilities are ignored, i.e. $u_i = 1$ for each i, then (2.20) reduces to

$$I_\alpha^\beta (P/Q) = \left[\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i^\beta} - 1 \right] (2^{\alpha-1} - 1)^{-1}, \; \alpha, \beta > 0, \; \alpha \neq 1 \tag{2.24}$$

which is non-additive directed divergence measure of order α and type β characterized by Patni and Jain (1977).

Further, if we put $\beta = 1$ in (2.24), we get

$$I_\alpha (P/Q;U) = \left[\frac{\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i} - 1 \right] (2^{\alpha-1} - 1)^{-1}, \quad \alpha \neq 1$$

which is a non-additive directed divergence measure of order α introduced by Nath (1972).

3. MEASURES OF 'USEFUL' INFORMATION IMPROVEMENT AND J-DIVERGENCE

Let $I_n^* : \Delta_n^* \times R_+^n \to R_+$, $n \geq 2$

where

$$\Delta_n^* = \left\{ (p_1, p_2, \cdots, p_n) / (r_1, r_2, \cdots, r_n); 0 < p_i, \; r_i \leq 1, \; \sum_{i=1}^{n} p_i \leq 1, \; \sum_{i=1}^{n} r_i \leq 1 \right\}$$

$R_+^n = \{(u_1, u_2, \cdots, u_n); \; u_i > 0 \text{ for every } i\}$ and $R_+ = (0, \infty)$ be a function satisfying the postulates 2.1 – 2.5 of the previous section. Then by following the same procedure, we have

$$I_{\alpha}^{\beta}(P/R;U)=\frac{\sum_{i=1}^{n}u_i^{\beta}p_i^{\beta}[(p_i/r_i)^{\alpha-1}-1]}{(2^{\alpha-1}-1)\sum_{i=1}^{n}p_i^{\beta}} \tag{3.1}$$

and

$$I_{\alpha}^{\beta}(P:Q\,;R\,;U)=I(P/Q\,;U)-I(P/R\,;U)$$

$$=\frac{\sum_{i=1}^{n}u_i^{\beta}p_i^{\beta}[(p_i/q_i)^{\alpha-1}-(p_i/r_i)^{\alpha-1}]}{(2^{\alpha-1}-1)\sum_{i=1}^{n}p_i^{\beta}} \tag{3.2}$$

We may call the measure (3.2) as the non-additive generalized measure of 'useful' information improvement of type α and degree β. The non-additive generalized measure of 'useful' $J-$divergence of type α and degree β is given by

$$J_{\alpha}^{\beta}(P/Q;U)=\frac{\sum_{i=1}^{n}u_i^{\beta}p_i^{\beta}[(p_i/q_i)^{\alpha-1}-1]}{(2^{\alpha-1}-1)\sum_{i=1}^{n}p_i^{\beta}}+\frac{\sum_{i=1}^{n}u_i^{\beta}q_i^{\beta}[(q_i/p_i)^{\alpha-1}-1]}{(2^{\alpha-1}-1)\sum_{i=1}^{n}q_i^{\beta}}$$

$$=\frac{1}{2^{\alpha-1}-1}\left[\frac{\sum_{i=1}^{n}u_i^{\beta}p_i^{\beta}[(p_i/q_i)^{\alpha-1}-1]}{\sum_{i=1}^{n}p_i^{\beta}}+\frac{\sum_{i=1}^{n}u_i^{\beta}q_i^{\beta}[(q_i/p_i)^{\alpha-1}-1]}{\sum_{i=1}^{n}q_i^{\beta}}\right] \tag{3.3}$$

Particular Cases:

Case 3.1) When $\beta=1$, and $\alpha\to 1$, (3.2) reduces to

$$I(P:Q:R;U)=\frac{\sum_{i=1}^{n}u_i p_i\log(r_i/q_i)}{\sum_{i=1}^{n}p_i} \tag{3.4}$$

which is (1.9).

Case 3.2) When $\beta = 1$, and $\alpha \to 1$,

$$J(P/Q\,;U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i / q_i)}{\sum_{i=1}^{n} p_i} + \frac{\sum_{i=1}^{n} u_i q_i \log(q_i / p_i)}{\sum_{i=1}^{n} q_i} \tag{3.5}$$

which is a 'useful' $J-$divergence measure.

Properties:

Next we study major properties satisfied by the non-additive generalized 'useful' relative information measure of order α and type β.

Property 3.1) Symmetry

The measure $I_\alpha^\beta(P/Q\,;U)$ is a symmetric function of its argument, that is

$$I_n^*(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, q_n\};\{u_1, u_2, \cdots, .u_n\})$$

$$= I_n^*(\{p_{a_1}, p_{a_2}, \cdots, p_{a_n}\}/\{q_{a_1}, q_{a_2}, \cdots, q_{a_n}\};\{u_{a_1}, u_{a_2}, \cdots, u_{a_n}\})$$

where $(a_1, a_2, \cdots, a_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$.

Property 3.2) Normality

Assuming $0\log 0 = 0$, we have

$$I^*(\{1,0\}/\{1/2,\ 1/2\};\{1,u\}) = 1$$

Property 3.3) Decisivity

$$I_\alpha^\beta(P/Q\,;U) = 0, \ \Leftrightarrow \ P = Q$$

Property 3.4) Continuity

$$I^*(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, q_n\};\{u_1, u_2, \cdots, .u_n\})$$

is a continuous function of its arguments.

Property 3.5) Expansibility

Assuming $0\log 0 = 0$, we have

$$I_n^*(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, .q_n\};\{u_1, u_2, \cdots, u_n\})$$

$$= I_{n+1}^*(\{0, p_1, p_2, \cdots, p_n\}/\{0, q_1, q_2, \cdots, q_n\};\{u, u_1, u_2, \cdots, u_n\})$$

Property 3.6) Non-additivity

The measure $I_\alpha^\beta(P/Q\,;U)$ satisfies the non-additivity as follows:

$$I_\alpha^\beta(P*P'/Q*Q'\,;U*U') = U'\,I_\alpha^\beta(P/Q;U) + U\,I_\alpha^\beta(P'/Q';U') + (2^{\alpha-1}-1)\,I_\alpha^\beta(P/Q;U)\,I_\alpha^\beta(P'/Q';U') \tag{3.6}$$

where $\alpha(\neq 1)$ and β are arbitrary constants and

$$\overline{U}' = \frac{\sum_{j=1}^{n}(u'_j p'_j)^\beta}{\sum_{j=1}^{n} p'^{\beta}_j} \quad \text{and} \quad \overline{U} = \frac{\sum_{i=1}^{n}(u_j p_j)^\beta}{\sum_{i=1}^{n} p_i^\beta}$$

$$P*P' = (p_1p'_1, p_2p'_2, \cdots, p_1p'_m, \cdots, p_np'_1, p_np'_2, \cdots, p_np'_m)$$

$$Q*Q' = (q_1q'_1, q_1q'_2, \cdots, q_1q'_m, \cdots, q_nq'_1, q_nq'_2, \cdots, q_nq'_m)$$

$$U*U' = (u_1u'_1, u_1u'_2, \cdots, u_1u'_m, \cdots, u_nu'_1, u_nu'_2, \cdots, u_nu'_m)$$

The above-mentioned properties can be verified easily.

Property 3.7) Recursivity or Branching Property

Let A_i, A_j be two events having probabilities p_i, p_j and utilities u_i, u_j respectively, then we define the utility of the compound event $A_i \cup A_j$ as

$$Ut(A_i \cup A_j) = \frac{u_ip_i + u_jp_j}{p_i + p_j} \tag{3.7}$$

Theorem 3.1:

$$I_\alpha^\beta(P/Q;U) = I_\alpha^\beta(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, .q_n\};\{u_1, u_2, \cdots, u_n\})$$

satisfies the following under the composition law (3.7):

$$\begin{aligned} I^*_{n+1}(\{p_1, p_2, \cdots, p_{n-1}, p', p''\}/\{q_1, q_2, \cdots, q_{n-1}, q', q''\};\{u_1, u_2, \cdots, u_{n-1}, u', u''\}) \\ = I^*_n(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, q_n\};\{u_1, u_2, \cdots, u_n\}) \\ + (p_n/q_n)^{\alpha-1} I_2(\{p'/p_n, p''/p_n\}/\{q'/q_n, q''/q_n\};\{u', u''\}) \end{aligned} \tag{3.8}$$

where $p_n = p' + p''$, $q_n = q' + q''$ and $u_n^\beta = \dfrac{(u'p')^\beta + (u''p'')^\beta}{(p')^\beta + (p'')^\beta}$

Proof: We have

$$I_n^*(\{p_1, p_2, \cdots, p_n\}/\{q_1, q_2, \cdots, q_n\};\{u_1, u_2, \cdots, u_n\})$$

$$= \frac{\sum_{i=1}^{n} u_i^\beta p_i^\beta [(p_i/q_i)^{\alpha-1} - 1]}{(2^{\alpha-1} - 1)\sum_{i=1}^{n} p_i^\beta}$$

and

$$I_{n+1}^*(\{p_1, p_2, \cdots, p_{n-1}, p', p''\}/\{q_1, q_2, \cdots, q_{n-1}, q', q''\};\{u_1, u_2, \cdots, u_{n-1}, u', u''\}$$

$$= \frac{\sum_{i=1}^{n-1} u_i^\beta p_i^\beta [(p_i/q_i)^{\alpha-1} - 1]}{(2^{\alpha-1} - 1)\sum_{i=1}^{n-1} p_i^\beta}$$

$$+ \frac{(u'p')^\beta [(p'/q')^{\alpha-1} - 1] + (u''p'')^\beta [(p''/q'')^{\alpha-1} - 1]}{(2^{\alpha-1} - 1)[(p')^\beta + (p'')^\beta]}$$

Now

$$I_{n+1}^* - I_n^* = \frac{(u'p')^\beta [(p'/q')^{\alpha-1} - 1] + (u''p'')^\beta [(p''/q'')^{\alpha-1} - 1]}{(2^{\alpha-1} - 1)[(p')^\beta + (p'')^\beta]}$$

$$- \frac{u_n^\beta [(p_n/q_n)^{\alpha-1} - 1]}{2^{\alpha-1} - 1}$$

$$= (2^{\alpha-1} - 1)^{-1} \left[\frac{(u'p')^\beta [(p'/q')^{\alpha-1} - 1] + (u''p'')^\beta [(p''/q'')^{\alpha-1} - 1]}{[(p')^\beta + (p'')^\beta]} \right.$$

$$\left. - \frac{(u'p')^\beta + (u''p'')^\beta}{(p')^\beta + (p'')^\beta} \{(p_n/q_n)^{\alpha-1} - 1\} \right]$$

$$= \frac{(2^{\alpha-1} - 1)^{-1}}{[(p')^\beta + (p'')^\beta]} \left[\frac{p_n^{\alpha+\beta-1}}{q_n^{\alpha-1}} . (u')^\beta (p'/p_n)^\beta \left\{ \left(\frac{p'/p_n}{q'/q_n} \right)^{\alpha-1} - 1 \right\} \right.$$

$$\left. + (u'')^\beta \left(\frac{p''}{p_n} \right)^\beta \left\{ \left(\frac{p''/p_n}{q''/q_n} \right)^{\alpha-1} - 1 \right\} \right]$$

$$= \left(\frac{p_n}{q_n}\right)^{\alpha-1} I_2\left[\left\{\frac{p'}{p_n}, \frac{p''}{p_n}\right\} / \left\{\frac{q'}{q_n}, \frac{q''}{q_n}\right\}; (u', u'')\right].$$

This completes the proof of theorem 3.1.

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